Quadratic and p-th variation of random signed Takagi–Landsberg bridges

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Abstract

We study random signed Takagi–Landsberg bridges with independent Rademacher Faber–Schauder coefficients. At index $H = 1/2$, we prove that every fixed deterministic refining sequence of partitions with vanishing mesh yields quadratic variation t, almost surely and uniformly in $t \in [0, 1]$. For partitions that need not be refining, the mesh condition $o(1/\log n)$ is sufficient, and its order is sharp. Our proofs use an operator representation of the quadratic sums and moment bounds for Rademacher chaos. We also show that asymmetric signs can destroy this invariance. At every index $H \neq 1/2$, we construct a deterministic refining sequence along which the critical p-power sums, with $p = 1/H$, have two distinct finite accumulation points almost surely. At $H = 1/4$, uniform thirds grids provide an explicit alternative to the dyadic limit. Thus $H = 1/2$ is the unique index in the symmetric family at which critical variation is invariant along every fixed deterministic refining sequence. Nevertheless the variation index equals $1/H$ almost surely across a large sequence of partitions.

1 Introduction

Paul Lévy [13] proved that Brownian motion has linear quadratic variation along every fixed deterministic refining sequence of partitions. More precisely, let B be standard Brownian motion and, for every n, write

$$\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}, \qquad n \in \mathbb{N},$$

for partitions of the time interval $[0, T]$. If (π_n) is any fixed deterministic refining sequence with vanishing mesh, then

$$\sum_{i=0}^{N_n - 1} (B_{t_{i+1}^n} - B_{t_i^n})^2 \longrightarrow T \qquad \text{almost surely.}$$

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Applying the terminal-time statement at rational times and using monotonicity yields the familiar pathwise formulation $\langle B \rangle_t = t$ simultaneously for all $t \in [0, T]$.

For Lévy's later systematic treatment of Brownian sample path properties, see his 1948 monograph [14]. Wiener [20] had earlier introduced a class of functions with bounded quadratic variation in his study of Fourier coefficients, although his notion of 2-variation differs from the limit along a single partition sequence used here. Protter [18, Chapter I, Theorem 28] gives a short, modern proof of Lévy's theorem via the backward martingale convergence theorem.

For Brownian motion, the condition that the partitions (π_n) are nested can be relaxed. The conclusion of L^2-convergence holds whenever the mesh sizes $\mathrm{mesh}(\pi_n)$ converge to zero as $n \uparrow \infty$. Dudley [8, p. 89] proved almost-sure convergence for arbitrary deterministic partitions satisfying $\mathrm{mesh}(\pi_n) = o(1/\log n)$. Fernández de la Vega [9] showed that this condition is best possible by constructing a partition sequence with $\mathrm{mesh}(\pi_n) = O(1/\log n)$ for which almost-sure convergence to the usual quadratic variation fails.

The order of the quantifiers in Lévy's theorem is essential. It says that for each fixed deterministic refining sequence there is a probability-one event on which convergence holds; it does not supply one event working for all partition sequences simultaneously. This distinction became especially important after Föllmer [10] defined quadratic variation along a prescribed partition sequence as a property of an individual path and derived the corresponding pathwise Itô formula. In that setting the limit can depend strongly on the chosen partitions, particularly when they are selected after the path is known. The resulting problem is to identify conditions on the path, on the partitions, or on their interaction that restore stability.

The possible dependence of quadratic variation on the choice of a partition sequence is extreme. Building on observations of Lévy and Freedman [11, pp. 47–48], Davis, Obłój, and Siorpaes [7, Theorem 7.4] proved that, for Brownian motion, suitable random refining partitions can realize any prescribed jointly measurable nondecreasing process starting from zero as a limit of quadratic sums. They also show in Proposition 2.3 that, for optional partitions whose oscillation along the path tends to zero almost surely, one can extract a subsequence along which the canonical semimartingale quadratic variation is recovered almost surely. This sharp contrast with the fixed-deterministic-sequence theorem is another reason to keep the quantifiers explicit.

Cont and Das give conditions for stability with respect to the partition choice. In [5, Definition 3.2], they introduce the notion of quadratic roughness, and their Theorem 4.2 shows that if a path is α-Hölder and quadratically rough along a balanced partition sequence with coarsening index $\beta \in (0, 1 \wedge 2\alpha)$, then its quadratic variation along that sequence exists and agrees with its dyadic quadratic variation. For Brownian motion, they prove in their Theorem 3.4 that, for each $0 < \beta < 1$, this roughness property holds almost surely along any fixed balanced sequence satisfying $(\log n)^2 \,\mathrm{mesh}(\pi_n) \to 0$.

In [4], Cont and Das construct nonuniform Haar and Faber–Schauder systems associated with finitely refining partitions. For a specific class of random Schauder expansions, they prove that the quadratic sums along the reference partition and every finitely refining coarsening converge to t in probability. If the reference partition and its coarsening are both balanced and complete refining, the corresponding quadratic variations exist and agree almost surely. They also exhibit a deterministic Schauder series for which a balanced finitely refining coarsening changes the limiting quadratic variation.

We study partition dependence for processes that connect Brownian bridges with classical fractal functions. Replacing the independent standard normal coefficients in the Lévy–Ciesielski expansion of a Brownian bridge by independent symmetric Rademacher variables gives a non-Gaussian process X with the same covariance, $\mathbb{E}[X_s X_t] = s \wedge t - st$. Its sample paths belong to the flexible class of Takagi functions introduced by Allaart [1] and studied further by Mishura and Schied [15]. Unlike Brownian paths, these paths are 1/2-Hölder continuous. A Brownian bridge inherits Lévy's theorem from Brownian motion by subtracting the finite-variation path $t \mapsto tB_1$. For the Rademacher bridge, this argument is unavailable. Nevertheless, we prove that every fixed deterministic refining sequence with vanishing mesh yields quadratic variation t, almost surely and uniformly in time.

We also examine the roles of refinement and symmetry. For partitions that need not be refining, the mesh condition $o(1/\log n)$ remains sufficient. At order $O(1/\log n)$, we construct one deterministic sequence along which the terminal quadratic sums have distinct lower and upper limits for every sign array. If the independent signs have a common nonzero mean, even refining sequences can give a limit different from the dyadic one or fail to give a limit at all.

Our second purpose is to study critical variation at other indices. Mishura and Schied [15] proved that every signed Takagi–Landsberg function of index $H \in (0, 1)$ has linear dyadic p-th variation at $p = 1/H$. For every $H \neq 1/2$, however, we construct a fixed deterministic refining sequence along which the terminal p-power sums have two distinct finite accumulation points almost surely. At $H = 1/4$, a separate calculation gives an explicit limit on uniform thirds grids. These results identify both the scope and the limitations of the quadratic invariance theorem.

The results of this paper were obtained independently of the very recent preprint by Cont [3], which establishes intrinsic quadratic roughness for the same Rademacher Faber–Schauder process and demonstrates failure of intrinsic roughness at fourth order. His concentration estimates also imply the sufficient logarithmic mesh condition considered here. Our additional contributions include quadratic invariance along every fixed deterministic refining sequence without a mesh-rate assumption, a matching sharpness construction for the logarithmic threshold, and explicit results for biased independent signs. At every $H \neq 1/2$, we construct a fixed deterministic refining sequence along which the critical sums have two distinct almost-sure accumulation points, extending the partition-dependence phenomenon beyond the fourth-order case. We also establish concentration of the critical sums around their expectations for arbitrary deterministic partitions under a mesh condition, yielding a first-moment characterization of convergence and a partition-independent variation index within this class of sequences.

Section 2 introduces the model and states the results. The proofs are collected in Section 3.

2 Main results

We first introduce the bridges and the notions of variation used below. We then state the quadratic invariance theorem, examine the effects of partition mesh and asymmetric signs, and turn to critical variation at $H \neq 1/2$.

2.1 The model and notions of variation

Let

$$e_{n,k}(t) = 2^{-n/2} e_{0,0}(2^n t - k), \qquad e_{0,0}(u) = \max\{0, u \wedge (1-u)\},$$

be the Faber–Schauder functions, where $n \geq 0$ and $0 \leq k < 2^n$. If $(\xi_{n,k})$ are independent standard normal random variables, then the series

$$B_t^\circ := \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \xi_{n,k} e_{n,k}(t)$$

converges uniformly in $t \in [0,1]$ almost surely, and B° is a standard Brownian bridge.

Let $\{\theta_{n,k}\}$ be independent Rademacher random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that

$$\mathbb{P}(\theta_{n,k} = 1) = \mathbb{P}(\theta_{n,k} = -1) = \frac{1}{2}.$$

For $H \in (0,1)$, we define

$$X_t^H := \sum_{n=0}^{\infty} 2^{n(1/2-H)} \sum_{k=0}^{2^n-1} \theta_{n,k} e_{n,k}(t), \qquad 0 \leq t \leq 1. \tag{2.1}$$

We write $X := X^{1/2}$. At level n, the sum in (2.1) has sup norm at most 2^{-nH-1}; hence the series converges uniformly for every sign array. Every realization is therefore continuous and vanishes at both endpoints. The functions in (2.1) belong to the flexible Takagi class introduced by Allaart [1]. Mishura and Schied [15] studied this subclass under the name signed Takagi–Landsberg functions. In the present paper, we emphasize the fact that X^H is a stochastic process and call it the *random signed Takagi–Landsberg bridge of index H*. We use “random signed” to mean that every tent sign is chosen independently, while we use “bridge” only for the endpoint pinning; the term does not assert that the law arises by conditioning another process.

Since the Rademacher and Gaussian coefficients have the same means and variances, the covariance at index 1/2 is

$$\mathbb{E}[X_s X_t] = s \wedge t - st, \tag{2.2}$$

which is the covariance kernel of a standard Brownian bridge. Thus X has the covariance of a Brownian bridge, although its law is not Gaussian.

To compare partition sequences, we use stopped power sums and their terminal values.

For a finite partition $\tau = \{0 = t_0 < \cdots < t_N = 1\}$, a continuous path x, and $p \geq 1$, we set

$$\operatorname{mesh}(\tau) := \max_{0 \leq i < N} (t_{i+1} - t_i)$$

and define

$$[x]^{(p)}_{(\tau)}(t) := \sum_{i=0}^{N-1} |x(t_{i+1} \wedge t) - x(t_i \wedge t)|^p, \qquad 0 \le t \le 1, \tag{2.3}$$

$$[x]^{(p)}_{(\tau)} := [x]^{(p)}_{(\tau)}(1) = \sum_{i=0}^{N-1} |x(t_{i+1}) - x(t_i)|^p. \tag{2.4}$$

When $p = 2$, we omit the superscript and write $[x]_{(\tau)}(t)$ and $[x]_{(\tau)}$ for the stopped and terminal quadratic sums, respectively.

Definition 2.1 (Continuous p-th variation)**.** *Let $\pi = (\pi_m)_{m\ge1}$ be a sequence of finite partitions with vanishing mesh. A continuous path x has* continuous p-th variation along π *if $[x]^{(p)}_{(\pi_m)}(t)$ converges uniformly in $t \in [0,1]$ to a continuous function.*

Remark 1. Another convention includes the entire increment whose left endpoint does not exceed t. We denote these sums by

$$V^{(p)}_\tau(x;t) := \sum_{\substack{0\le i<N \\ t_i\le t}} |x(t_{i+1}) - x(t_i)|^p, \qquad 0 \le t \le 1,$$

and write $V_\tau(x;t) := V^{(2)}_\tau(x;t)$ when $p = 2$. Writing $\omega_x(\delta) := \sup_{|u-v|\le\delta} |x(u) - x(v)|$, we have

$$\sup_{0\le t\le1} \left| V^{(p)}_\tau(x;t) - [x]^{(p)}_{(\tau)}(t) \right| \le \omega_x(\mathrm{mesh}(\tau))^p. \tag{2.5}$$

Indeed, only the interval beginning at $\max(\tau \cap [0,t])$ can contribute to the difference, and both corresponding terms lie between zero and the right-hand side. Thus the conventions have the same limits as the mesh vanishes. The successor sums are nondecreasing in t; consequently, pointwise convergence to a continuous limit also implies uniform convergence. We will use this observation to pass from rational times to all times.

Throughout, partition sequences are deterministic and have vanishing mesh, unless stated otherwise. We write $\pi = (\pi_m)_{m\ge1}$, where

$$\pi_m = \{0 = t^m_0 < t^m_1 < \cdots < t^m_{N_m} = 1\}.$$

In particular, the partition points do not depend on the path.

Definition 2.2 (Refining partitions)**.** *A sequence π is* refining *if $\pi_m \subseteq \pi_{m+1}$ for every m. It is* finitely refining *if, in addition, there is a finite constant M such that each interval of π_m contains at most M new points of π_{m+1}, uniformly in m.*

We denote the dyadic sequence by $\mathbb{T} = (\mathbb{T}_m)$, where

$$\mathbb{T}_m := \{k2^{-m} : 0 \le k \le 2^m\}.$$

These partitions provide our benchmark. Mishura and Schied [15, corrected arXiv version, Theorem 2.1] proved that, for $p = 1/H$ and every choice of the signs,

$$[X^H]^{(p)}_{(\mathbb{T}_n)}(t) \longrightarrow tC_{\mathrm{dyad}}(H), \qquad C_{\mathrm{dyad}}(H) := 2^{1-p}\mathbb{E}|Z_H|^p, \tag{2.6}$$

uniformly in $t \in [0,1]$, where

$$Z_H := \sum_{m=0}^{\infty} 2^{m(H-1)} Y_m$$

and (Y_m) is an auxiliary independent Rademacher sequence. The factor 2^{1-p} is included in the corrected arXiv version; it is missing from the published theorem statement and the corresponding terminal-time calculation. We also recover the terminal limit in Lemma 3.5 below.

A variation index can be defined even when the critical power sums do not converge; see, e.g., [6, 12].

Definition 2.3 (Variation index)**.** *For $x \in C([0,1])$, the* variation index along π *is*

$$p^\pi(x) := \inf\left\{ p \geq 1 : \limsup_{m\to\infty} [x]^{(p)}_{(\pi_m)} = 0 \right\}, \tag{2.7}$$

with $\inf \varnothing = \infty$*.*

For $1 \leq q < r$, continuity gives

$$[x]^{(r)}_{(\pi_m)} \leq \omega_x(\mathrm{mesh}(\pi_m))^{r-q} [x]^{(q)}_{(\pi_m)}.$$

It follows that

$$\limsup_{m\to\infty} [x]^{(q)}_{(\pi_m)} = \begin{cases} 0, & q > p^\pi(x), \\ \infty, & 1 \leq q < p^\pi(x). \end{cases} \tag{2.8}$$

Since $C_{\mathrm{dyad}}(H)$ is finite and positive, (2.6) shows that $p^{\mathbb{T}}(X^H) = 1/H$ for every sign array. We therefore call $p = 1/H$ the *critical exponent.*

2.2 Quadratic variation at index $H = 1/2$

At $H = 1/2$, the process has the covariance of a Brownian bridge and 1/2-Hölder continuous paths. Our first theorem shows that it also shares the Brownian bridge's quadratic variation along every fixed refining sequence.

Theorem 2.4 (Quadratic variation along refining partitions)**.** *For every fixed deterministic refining sequence π, there is an event Ω_π with $\mathbb{P}(\Omega_\pi) = 1$ such that, for every $\omega \in \Omega_\pi$,*

$$\lim_{m\to\infty} \sup_{t\in[0,1]} \left| [X(\omega)]_{(\pi_m)}(t) - t \right| = 0.$$

Remark 2. The event Ω_π may depend on the fixed sequence π. The theorem does not assert that one probability-one event works for all partition sequences simultaneously. This is the same order of quantifiers as in Lévy's theorem and permits the path-dependent constructions discussed in the introduction.

Without refinement, a sufficiently fast mesh decay still guarantees convergence. The next result gives the analogue of Dudley's sufficient condition and shows that its order is sharp as Fernández de la Vega [9] showed it to be for Brownian motion. In the sharpness assertion, failure occurs for every sign array.

Theorem 2.5. *Let $X = X^{1/2}$ be the random signed Takagi–Landsberg bridge.*

(a) *Let (π_n) be a deterministic sequence of partitions, not necessarily refining. If*

$$\operatorname{mesh}(\pi_n) = o\Big(\frac{1}{\log n}\Big),$$

then

$$\sup_{t\in[0,1]} \Big|[X]_{(\pi_n)}(t) - t\Big| \longrightarrow 0 \qquad \text{almost surely.}$$

(b) *Set*

$$\gamma := \frac{4+\sqrt{2}}{24} = \frac{2+1/\sqrt{2}}{12}.$$

There exists a fixed deterministic, nonrefining sequence $(\widehat{\pi}_n)$ such that

$$\operatorname{mesh}(\widehat{\pi}_n) = O\Big(\frac{1}{\log n}\Big)$$

and, for every sign array,

$$\liminf_{n\to\infty} [X]_{(\widehat{\pi}_n)} = 1-\gamma, \qquad \limsup_{n\to\infty} [X]_{(\widehat{\pi}_n)} = 1+\gamma.$$

In particular, the little-o condition in the first assertion cannot be replaced by a big-O condition.

The symmetry assumption also matters. The next theorem shows that a common nonzero mean of the signs can change the quadratic variation along refining sequences. Uniform thirds grids yield a limit larger than the dyadic one, whereas triadic grids yield two distinct subsequential limits.

Theorem 2.6. *Fix any $\nu \in (0,1)\setminus\{1/2\}$, and let $\{\theta^{(\nu)}_{m,k}\}$ be independent random variables with*

$$\mathbb{P}\Big(\theta^{(\nu)}_{m,k} = 1\Big) = \nu, \qquad \mathbb{P}\Big(\theta^{(\nu)}_{m,k} = -1\Big) = 1-\nu.$$

Define

$$Y^{(\nu)}_t := \sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1} \theta^{(\nu)}_{m,k} e_{m,k}(t), \qquad 0 \le t \le 1,$$

and put $b := 2\nu - 1$. Consider the uniform thirds grids and the triadic grids

$$\mathbb{U}_n := \left\{\frac{j}{3\cdot 2^n} : 0 \le j \le 3\cdot 2^n\right\}, \qquad \mathbb{T}^{(3)}_n := \left\{\frac{j}{3^n} : 0 \le j \le 3^n\right\}.$$

Then

$$[Y^{(\nu)}]_{(\mathbb{U}_n)} \longrightarrow 1 + \frac{6+8\sqrt{2}}{9} b^2 \qquad \text{almost surely.} \tag{2.9}$$

In particular, this limit is strictly larger than the dyadic limit of 1.

Along the triadic grids, the quadratic sums do not converge almost surely. More precisely, there are deterministic subsequences (n_j) and (m_j) such that, almost surely,

$$[Y^{(\nu)}]_{(\mathbb{T}^{(3)}_{n_j})} \longrightarrow 1 - \frac{4}{9} b^2, \tag{2.10}$$

$$[Y^{(\nu)}]_{(\mathbb{T}^{(3)}_{m_j})} \longrightarrow 1 - \frac{28}{81} b^2. \tag{2.11}$$

The two limits differ by $8b^2/81 > 0$. *By contrast, for every realization of the signs,*

$$[Y^{(\nu)}]_{(\mathbb{T}_n)} = 1 - 2^{-n}, \tag{2.12}$$

and hence the dyadic quadratic sums converge to 1.

2.3 Critical variation at indices $H \neq 1/2$

For $H \neq 1/2$, the critical exponent $p = 1/H$ differs from two. The next theorem shows that the corresponding variation is not invariant under the choice of a fixed deterministic refining sequence.

Theorem 2.7. *Let* $H \in (0,1) \setminus \{1/2\}$ *and* $p = 1/H$. *There is a fixed deterministic refining sequence* $\pi^H = (\pi_n^H)$ *with vanishing mesh such that*

$$[X^H]^{(p)}_{(\pi_n^H)}$$

fails to converge almost surely. Hence X^H *does not admit continuous critical* p*-th variation along* π^H.

More precisely, for some integer $r = r(H) \geq 1$, *let*

$$\mathbb{V}_n^{(r)} := \mathbb{T}_n \cup \left\{2^{-n}(k + 2^{-r}) : 0 \leq k < 2^n\right\}. \tag{2.13}$$

There is a deterministic constant $C_r(H)$ *such that, almost surely,*

$$[X^H]^{(p)}_{(\mathbb{V}_n^{(r)})} \longrightarrow C_r(H),$$

and

$$C_r(H) > C_{\mathrm{dyad}}(H) \quad \text{if } H < \frac{1}{2}, \qquad C_r(H) < C_{\mathrm{dyad}}(H) \quad \text{if } H > \frac{1}{2}.$$

With such an r *fixed, one may take, for* $j \geq 1$,

$$\pi^H_{2j-1} := \mathbb{V}^{(r)}_{jr}, \qquad \pi^H_{2j} := \mathbb{T}_{(j+1)r}. \tag{2.14}$$

The odd terminal sums then converge almost surely to $C_r(H)$ *and the even terminal sums to* $C_{\mathrm{dyad}}(H)$.

Thus $H = 1/2$ is the only index in this family at which critical variation is stable along every fixed deterministic refining sequence. The theorem uses a split at an H-dependent dyadic fraction. At $H = 1/4$, the following example identifies a competing limit explicitly on the simpler uniform thirds grids.

Proposition 2.8 (Explicit fourth-variation example)**.** *Let* $H = 1/4$, *and let* $(\mathbb{U}_n)$ *be the uniform thirds grids defined in Theorem 2.6. Then*

$$[X^{1/4}]^{(4)}_{(\mathbb{U}_n)} \longrightarrow C_{\mathrm{thirds}} := \frac{1301 + 1020\sqrt{2}}{1323} \qquad \textit{almost surely.} \tag{2.15}$$

This differs from the dyadic value:

$$C_{\mathrm{dyad}}(1/4) = \frac{13 + 12\sqrt{2}}{49}, \qquad C_{\mathrm{thirds}} - C_{\mathrm{dyad}}(1/4) = \frac{950 + 696\sqrt{2}}{1323} > 0. \tag{2.16}$$

Remark 3. Proposition 2.8 makes the partition dependence concrete at $H = 1/4$: the dyadic and uniform thirds grids yield different fourth-variation constants. We do not claim that the thirds-grid constant differs from $C_{\mathrm{dyad}}(H)$ for every $H \neq 1/2$; Theorem 2.7 establishes the general result through the H-dependent split grids.

2.4 Invariance of variation index across partition

Theorem 2.7 shows that the value of the critical variation depends on the partition sequence when $H \neq 1/2$. The results of this subsection show that the critical generalized p-th variations are uniformly bounded above over all partitions, for every sign array. For each fixed deterministic sequence satisfying (2.19), their fluctuations around their expectations vanish almost surely, their lower limit is positive almost surely, and the variation index is $1/H$. Under this mesh condition, convergence of the terminal critical sums is therefore entirely a first-moment property. The probability-one event may depend on the prescribed sequence; no balance assumption is needed.

Put

$$\kappa_H := \min\{1,\ 2-2H\} \in (0,1], \qquad H \in (0,1). \tag{2.17}$$

Proposition 2.9 (Deterministic upper bound)**.** *Let $H \in (0,1)$, $p = 1/H$, and let L_H be the constant of Lemma 3.3. For every finite partition τ of $[0,1]$ and every sign array,*

$$[X^H]^{(p)}_{(\tau)} \leq L_H^p. \tag{2.18}$$

Consequently, for every partition sequence π with vanishing mesh and every sign array, $[X^H]^{(q)}_{(\pi_n)} \to 0$ for all $q > p$, and $p^\pi(X^H) \leq 1/H$.

Theorem 2.10 (Asymptotic determinism of the critical sums)**.** *Let $H \in (0,1)\setminus\{1/2\}$, $p = 1/H$, and let π be a fixed deterministic partition sequence with*

$$\mathrm{mesh}(\pi_n) = o\Big((\log n)^{-1/\kappa_H}\Big). \tag{2.19}$$

Then there is a constant $c_H > 0$, depending only on H, such that for all sufficiently large n,

$$c_H \leq \mathbb{E}\,[X^H]^{(p)}_{(\pi_n)} \leq L_H^p, \tag{2.20}$$

and

$$[X^H]^{(p)}_{(\pi_n)} - \mathbb{E}\,[X^H]^{(p)}_{(\pi_n)} \longrightarrow 0 \qquad \textit{almost surely.} \tag{2.21}$$

In particular, almost surely,

$$c_H \leq \liminf_{n\to\infty} [X^H]^{(p)}_{(\pi_n)} \leq \limsup_{n\to\infty} [X^H]^{(p)}_{(\pi_n)} \leq L_H^p, \tag{2.22}$$

and $p^\pi(X^H) = 1/H$.

For $H < 1/2$ the mesh condition (2.19) is Dudley's condition $\mathrm{mesh}(\pi_n) = o(1/\log n)$ of Theorem 2.5(a). For $H > 1/2$, the bounded-differences estimate below decays as $\mathrm{mesh}(\pi_n)^{2-2H}$, which gives the stronger sufficient condition $\mathrm{mesh}(\pi_n) = o((\log n)^{-1/(2-2H)})$.

Corollary 2.11 (Convergence is a first-moment property)**.** *Under the assumptions of Theorem 2.10, the following are equivalent:*

(i) $[X^H]^{(p)}_{(\pi_n)}$ *converges almost surely;*

(ii) $[X^H]^{(p)}_{(\pi_n)}$ *converges in probability;*

(iii) *the deterministic sequence* $\mathbb{E}\,[X^H]^{(p)}_{(\pi_n)}$ *converges.*

When they hold, the almost-sure limit is $\lim_n \mathbb{E}[X^H]^{(p)}_{(\pi_n)}$.

Thus the limits in Theorem 2.7 and Proposition 2.8 can be identified through first moments. For the sequence (2.14),

$$\operatorname{mesh}(\pi^H_{2j-1}) = (1-2^{-r})2^{-jr}, \qquad \operatorname{mesh}(\pi^H_{2j}) = 2^{-(j+1)r},$$

so (2.19) holds. Its terminal sums fail to converge because their expectations converge along the odd and even subsequences to the distinct constants $C_r(H)$ and $C_{\mathrm{dyad}}(H)$, respectively. Similarly, the thirds grids satisfy the mesh condition. Conversely, for each fixed deterministic sequence satisfying (2.19), (2.22) places all subsequential limits in $[c_H, L^p_H]$ almost surely.

The reciprocal H of the variation index $1/H$ is the roughness exponent in the convention of Han and Schied [12]. It is also the Besov smoothness exponent $1/p$ appearing in [3, Lemma 3.2]. That lemma assumes intrinsic p-roughness, a property not established by the present concentration theorem. Here the conclusion is that the variation index equals $1/H$ almost surely along each fixed deterministic sequence satisfying (2.19).

3 Proofs

We now prove the results stated in Section 2.

3.1 Probabilistic and regularity estimates

We first record the moment inequalities used in the refining-partition argument and a deterministic regularity estimate.

Let $\theta_1, \theta_2, \dots$ be independent Rademacher variables. A *finite homogeneous Rademacher chaos of order two* is a random variable of the form

$$F = \sum_{1\le i<j\le N} a_{ij}\theta_i\theta_j, \qquad a_{ij}\in\mathbb{R}. \tag{3.1}$$

Thus every monomial contains exactly two distinct Rademacher variables; in particular, there are no constant, linear, or diagonal terms. We use the same terminology for an L^2 limit of finite chaoses of this form, equivalently for an expansion

$$F = \sum_{i<j} a_{ij}\theta_i\theta_j, \qquad \sum_{i<j} a_{ij}^2 < \infty,$$

where the series converges in L^2.

Lemma 3.1 (Rademacher hypercontractivity)**.** *Let F be a homogeneous Rademacher chaos of order two. Then*

$$\|F\|_{L^4} \le 3\|F\|_{L^2}.$$

For a finite chaos, this is the $q = 4$, $d = 2$ case of the Bonami–Beckner hypercontractive inequality. O'Donnell [17, Theorem 9.21] proves the more general estimate

$$\|F\|_{L^q} \le (q-1)^{d/2}\|F\|_{L^2}, \qquad q \ge 2,$$

for every Rademacher polynomial of degree at most d. O'Donnell [17, Section 9.1] also gives a direct proof of the $q = 4$ case under the name *Bonami Lemma.*

For an L^2 limit of finite chaoses, we apply the finite inequality to the difference of two finite truncations. The truncations are then Cauchy in L^4, and passage to the limit yields the stated inequality.

A uniform subgaussian bound alone does not give the L^4–L^2 estimate in Lemma 3.1 for centered quadratic polynomials with diagonal terms. For example, let J_q be Bernoulli with parameter q, let R be an independent Rademacher variable, and set $\zeta = R\sqrt{1 + J_q - q}$. Then ζ is centered, has variance one, and is bounded by $\sqrt{2}$, whereas

$$\frac{\|\zeta^2 - 1\|_{L^4}}{\|\zeta^2 - 1\|_{L^2}} \sim q^{-1/4} \qquad (q \downarrow 0).$$

For the extension to general subgaussian coefficients, we instead use the operator moment bound (3.18) below.

Lemma 3.2 (Maximal moment inequality)**.** *Let $D_1, D_2, \ldots$ be arbitrary random variables and let $w_1, w_2, \ldots \ge 0$. Suppose that, for some $C < \infty$ and every $p \le q$,*

$$\mathbb{E}\left|\sum_{j=p}^{q} D_j\right|^4 \le C\left(\sum_{j=p}^{q} w_j\right)^2.$$

Then a constant C' depending only on C exists such that

$$\mathbb{E}\max_{p\le r\le q}\left|\sum_{j=p}^{r} D_j\right|^4 \le C'\left(\sum_{j=p}^{q} w_j\right)^2.$$

Móricz, Serfling, and Stout [16, Theorem 3.1] prove the required maximal power-moment inequality for arbitrary random variables. To obtain the displayed form, we take their moment order to be 4, their power exponent to be 2, and

$$g(p,q) = \sqrt{C}\sum_{j=p}^{q} w_j.$$

This control is additive and hence superadditive, so the cited theorem requires neither independence nor a martingale structure in this application. Its power exponent 2 is strictly greater than one, as required.

We also record a uniform deterministic regularity estimate [15].

Lemma 3.3 (Uniform Hölder estimate)**.** *For every $H \in (0,1)$ there is a finite constant L_H, independent of the sign array, such that the function X^H defined by* (2.1) *satisfies*

$$|X_t^H - X_s^H| \le L_H|t-s|^H, \qquad s,t \in [0,1]. \tag{3.2}$$

Proof. For $s = t$, both sides of (3.2) vanish. We may therefore assume $s \neq t$, set $\delta = |t - s| > 0$, and choose the integer $N \geq 0$ so that $2^{-(N+1)} < \delta \leq 2^{-N}$. The level-$m$ term of (2.1) has Lipschitz constant at most $2^{m(1-H)}$ and sup norm at most 2^{-mH-1}. Consequently,

$$|X_t^H - X_s^H| \leq \delta \sum_{m<N} 2^{m(1-H)} + \sum_{m \geq N} 2^{-mH}.$$

Each geometric sum is bounded by a constant depending only on H times δ^H. Hence (3.2) follows. □

3.2 Quadratic variation at index $H = 1/2$

We express every quadratic sum in the original dyadic Haar basis. This keeps the independent Rademacher coefficients fixed as the partition changes. A change to a partition-dependent Schauder basis would generally destroy their independence: for independent Rademacher variables θ_1, θ_2, the orthogonal transform

$$\eta_1 = \frac{\theta_1 + \theta_2}{\sqrt{2}}, \qquad \eta_2 = \frac{\theta_1 - \theta_2}{\sqrt{2}}$$

has uncorrelated coordinates of variance one, but $\eta_1 \eta_2 = 0$ almost surely. Coefficient transformations between partition-dependent Schauder systems are studied in [4, Section 6.1].

For a finite partition ρ of $[0, t]$, with $0 < t \leq 1$, let $I \in \rho$ denote its adjacent half-open intervals. We extend the terminal-sum notation to this interval and set

$$[X]_{(\rho)} := \sum_{(u,v] \in \rho} (X_v - X_u)^2, \qquad s(\rho) := \sum_{(u,v] \in \rho} (v - u)^2, \qquad Z_\rho := [X]_{(\rho)} - t + s(\rho). \tag{3.3}$$

Lemma 3.4 (Quadratic-sum energy bounds)**.** *For every finite partition ρ of $[0, t]$, Z_ρ is a homogeneous Rademacher chaos of order two, and*

$$\mathbb{E}\Big([X]_{(\rho)}\Big) = t - s(\rho), \qquad \mathbb{E}|Z_\rho|^2 \leq 2s(\rho), \qquad \mathbb{E}|Z_\rho|^4 \leq 324 s(\rho)^2. \tag{3.4}$$

If ρ and σ are partitions of the same interval $[0, t]$, and $\tau = \rho \cup \sigma$ is their common refinement, then

$$\mathbb{E}|Z_\rho - Z_\sigma|^2 \leq 2\Big(s(\rho) + s(\sigma) - 2s(\tau)\Big), \tag{3.5}$$

$$\mathbb{E}|Z_\rho - Z_\sigma|^4 \leq 324\Big(s(\rho) + s(\sigma) - 2s(\tau)\Big)^2. \tag{3.6}$$

In particular, if σ refines ρ, then

$$\mathbb{E}|Z_\rho - Z_\sigma|^2 \leq 2\Big(s(\rho) - s(\sigma)\Big), \tag{3.7}$$

$$\mathbb{E}|Z_\rho - Z_\sigma|^4 \leq 324\Big(s(\rho) - s(\sigma)\Big)^2. \tag{3.8}$$

Proof. Let $h_{n,k} = e'_{n,k}$ almost everywhere. These Haar functions form an orthonormal basis of

$$H_0 := \left\{ f \in L^2[0,1] : \int_0^1 f(u)\,du = 0 \right\};$$

adjoining the constant function $\mathbf{1}_{[0,1]}$ gives an orthonormal basis of $L^2[0,1]$. Let P be the orthogonal projection onto H_0, so that $Pf = f - (\int_0^1 f)\mathbf{1}_{[0,1]}$. For $I = (u, v]$,

$$X_v - X_u = \sum_{n,k} \theta_{n,k} \langle \mathbf{1}_I, h_{n,k} \rangle \quad \text{in } L^2(\Omega). \tag{3.9}$$

All inner products below are in $L^2[0,1]$.

Define the positive finite-rank operators

$$K_\rho := \sum_{I \in \rho} \mathbf{1}_I \otimes \mathbf{1}_I, \qquad A_\rho := P K_\rho P = \sum_{I \in \rho} P\mathbf{1}_I \otimes P\mathbf{1}_I,$$

where $(f \otimes g)v = \langle v, g \rangle f$. Their trace and Hilbert–Schmidt norm satisfy

$$\operatorname{tr}(A_\rho) = \sum_{I \in \rho} (|I| - |I|^2) = t - s(\rho), \tag{3.10}$$

$$\|K_\rho\|_{\mathrm{HS}}^2 = \sum_{I \in \rho} |I|^2 = s(\rho). \tag{3.11}$$

Here $|I| = v - u$ for $I = (u, v]$. The second identity follows because the nonzero eigenvalues of K_ρ are the interval lengths $|I|$.

To justify the quadratic expansion, set

$$\mathcal{I} := \{(n,k) : n \geq 0,\ 0 \leq k < 2^n\}, \qquad \mathcal{I}_N := \{(n,k) \in \mathcal{I} : n < N\},$$

and order $\mathcal{I}$ lexicographically. We write θ_α, e_α, and h_α for the corresponding signs, tents, and Haar functions, and P_N for the orthogonal projection onto their span with $\alpha \in \mathcal{I}_N$. The truncated series

$$X^{[N]} := \sum_{\alpha \in \mathcal{I}_N} \theta_\alpha e_\alpha$$

is the linear interpolation of X on $\mathbb{T}_N$ and converges uniformly to X. With $(A_\rho)_{\alpha\beta} := \langle h_\alpha, A_\rho h_\beta \rangle$, its quadratic sum is

$$[X^{[N]}]_{(\rho)} = \sum_{\alpha,\beta \in \mathcal{I}_N} (A_\rho)_{\alpha\beta} \theta_\alpha \theta_\beta.$$

Since $\theta_\alpha^2 = 1$, the diagonal terms tend to $\operatorname{tr}(A_\rho)$. The products $\theta_\alpha\theta_\beta$, $\alpha < \beta$, are orthonormal in $L^2(\Omega)$, and A_ρ is Hilbert–Schmidt. Thus the off-diagonal terms converge in L^2, giving

$$[X]_{(\rho)} = \operatorname{tr}(A_\rho) + 2 \sum_{\alpha < \beta} (A_\rho)_{\alpha\beta} \theta_\alpha \theta_\beta. \tag{3.12}$$

In particular,

$$Z_\rho = 2 \sum_{\alpha < \beta} (A_\rho)_{\alpha\beta} \theta_\alpha \theta_\beta \quad \text{in } L^2(\Omega). \tag{3.13}$$

For another partition σ of $[0,t]$, the nonempty intersections of intervals of ρ and σ are the intervals of τ. Hence

$$\operatorname{tr}(K_\rho K_\sigma) = \sum_{I\in\rho, J\in\sigma} |I \cap J|^2 = s(\tau), \tag{3.14}$$

and therefore

$$\|K_\rho - K_\sigma\|_{\mathrm{HS}}^2 = s(\rho) + s(\sigma) - 2s(\tau). \tag{3.15}$$

Compression by P cannot increase the Hilbert–Schmidt norm. Applying orthogonality to (3.13), we obtain

$$\begin{aligned}\mathbb{E}|Z_\rho - Z_\sigma|^2 &= 4\sum_{\alpha<\beta} |(A_\rho - A_\sigma)_{\alpha\beta}|^2 \\ &\le 2\|A_\rho - A_\sigma\|_{\mathrm{HS}}^2 \le 2\Big(s(\rho) + s(\sigma) - 2s(\tau)\Big). \end{aligned}\tag{3.16}$$

The same calculation with A_ρ alone gives $\mathbb{E}|Z_\rho|^2 \le 2s(\rho)$. Lemma 3.1 gives all the fourth-moment bounds, since $3^4 \cdot 2^2 = 324$. If σ refines ρ, then $\tau = \sigma$, yielding the final assertions. □

Remark 4 (Subgaussian coefficients). The mean and moment bounds above extend, with constants depending only on K, to

$$\widetilde{X}_t := \sum_{n,k} \zeta_{n,k} e_{n,k}(t), \qquad 0 \le t \le 1, \tag{3.17}$$

where the coefficients are independent, centered, have variance one, and satisfy $\sup_{n,k} \|\zeta_{n,k}\|_{\psi_2} \le K < \infty$. Here $\|\zeta\|_{\psi_2} := \inf\{a > 0 : \mathbb{E}\exp(\zeta^2/a^2) \le 2\}$. The subgaussian tail bound and a union bound give $\max_{k<2^n} |\zeta_{n,k}| = O(\sqrt{n+1})$ almost surely, so the series converges uniformly and defines a continuous bridge.

For a finite coefficient vector ζ and a symmetric matrix B, integration of the Hanson–Wright inequality [19, Theorem 1.1] gives

$$\Big\|\zeta^{\mathsf{T}} B\zeta - \operatorname{tr} B\Big\|_{L^4} \le C_K \|B\|_{\mathrm{HS}}. \tag{3.18}$$

Indeed, the tail is bounded by $2\exp[-c_K \min(u^2/\|B\|_{\mathrm{HS}}^2, u/\|B\|_{\mathrm{op}})]$, and $\|B\|_{\mathrm{op}} \le \|B\|_{\mathrm{HS}}$. Applying (3.18) to differences of finite Haar compressions permits passage to the limit in L^4. Thus

$$\widetilde{Z}_\rho := [\widetilde{X}]_{(\rho)} - t + s(\rho) = 2\sum_{\alpha<\beta} (A_\rho)_{\alpha\beta}\zeta_\alpha\zeta_\beta + \sum_\alpha (A_\rho)_{\alpha\alpha}(\zeta_\alpha^2 - 1). \tag{3.19}$$

The diagonal term is generally random. In particular, this is not a Rademacher chaos, and the constants in Lemma 3.4 need not remain 2 and 324. Nevertheless,

$$\|\widetilde{Z}_\rho - \widetilde{Z}_\sigma\|_{L^4} \le C_K \|A_\rho - A_\sigma\|_{\mathrm{HS}} \le C_K \Big(s(\rho) + s(\sigma) - 2s(\rho \cup \sigma)\Big)^{1/2}.$$

The same bound with a single partition proves $\|\widetilde{Z}_\rho\|_{L^4} \le C_K s(\rho)^{1/2}$. Consequently the proof of Theorem 2.4 below applies to $\widetilde{X}$ as well. This includes standard normal coefficients and the bounded variance-one laws Uniform$(-\sqrt{3}, \sqrt{3})$, $\sqrt{20}(\mathrm{Beta}(2,2) - 1/2)$, and $\sqrt{8}(\mathrm{Beta}(1/2,1/2) - 1/2)$. Related Schauder constructions with non-Gaussian coefficients appear in [2].

Proof of Theorem 2.4. Fix $t \in (0,1]$ and restrict the partitions to $[0,t]$ by setting

$$\rho_m(t) := (\pi_m \cap [0,t]) \cup \{t\}, \qquad s_m := s(\rho_m(t)), \qquad Z_m := Z_{\rho_m(t)}.$$

Then $[X]_{(\rho_m(t))} = [X]_{(\pi_m)}(t)$, and $(\rho_m(t))$ is refining. Consequently, s_m is nonincreasing and

$$0 \le s_m \le t \ \mathrm{mesh}(\pi_m) \longrightarrow 0. \tag{3.20}$$

Set $D_j = Z_j - Z_{j+1}$ and $w_j = s_j - s_{j+1}$. By (3.8), for $a \le b$,

$$\mathbb{E}\left|\sum_{j=a}^{b} D_j\right|^4 \le 324\left(\sum_{j=a}^{b} w_j\right)^2.$$

Lemma 3.2 and monotone convergence therefore give

$$\mathbb{E}\sup_{m\ge a}|Z_m - Z_a|^4 \le Cs_a^2.$$

Together with (3.4) and $(u+v)^4 \le 8(u^4+v^4)$ for $u, v \ge 0$, this yields

$$\mathbb{E}\sup_{m\ge a}|Z_m|^4 \le C's_a^2 \longrightarrow 0. \tag{3.21}$$

Fatou's lemma now implies $\limsup_m |Z_m| = 0$ almost surely. Since $[X]_{(\pi_m)}(t) = Z_m + t - s_m$, we obtain

$$[X]_{(\pi_m)}(t) \longrightarrow t \qquad \text{almost surely.} \tag{3.22}$$

At $t = 0$, this holds by definition.

Intersect the probability-one events in (3.22) over rational $t \in [0,1]$. On this event, (2.5) implies that the nondecreasing functions $V_{\pi_m}(X;t)$ converge to t at every rational time. For each integer $k \ge 1$, monotonicity between the points j/k, $0 \le j \le k$, gives

$$\sup_{t\in[0,1]}|V_{\pi_m}(X;t) - t| \le \max_{0\le j\le k}|V_{\pi_m}(X;j/k) - j/k| + \frac{1}{k}.$$

First let $m \to \infty$ and then $k \to \infty$. The successor sums converge uniformly to t, and (2.5) transfers this convergence to the stopped sums. This proves the theorem. □

Proof of Theorem 2.5. (a) We first derive a concentration bound for a finite partition ρ of $[0,t]$, where $0 < t \le 1$. The normalized indicators $|I|^{-1/2}\mathbf{1}_I$, $I \in \rho$, are eigenvectors of K_ρ with eigenvalues $|I|$. Together with orthogonal compression, this gives

$$\|A_\rho\|_{\mathrm{op}} \le \|K_\rho\|_{\mathrm{op}} = \mathrm{mesh}(\rho), \qquad \|A_\rho\|_{\mathrm{HS}}^2 \le s(\rho) \le t\,\mathrm{mesh}(\rho). \tag{3.23}$$

Let $A_{\rho,N} = P_N A_\rho P_N$ and retain the truncation $X^{[N]}$ from the proof of Lemma 3.4. Its quadratic sum is the quadratic form associated with $A_{\rho,N}$ and the independent Rademacher coefficients $(\theta_\alpha)_{\alpha\in\mathcal{I}_N}$. The Hanson–Wright inequality [19, Theorem 1.1] therefore yields a universal constant $c > 0$ such that, for every $u > 0$,

$$\mathbb{P}\left(\left|[X^{[N]}]_{(\rho)} - \mathrm{tr}(A_{\rho,N})\right| > u\right) \le 2\exp\left(-c\min\left\{\frac{u^2}{s(\rho)}, \frac{u}{\mathrm{mesh}(\rho)}\right\}\right). \tag{3.24}$$

For each fixed ρ, uniform convergence of $X^{[N]}$ gives $[X^{[N]}]_{(\rho)} \to [X]_{(\rho)}$ for every sign array, and trace-norm convergence gives $\operatorname{tr}(A_{\rho,N}) \to \operatorname{tr}(A_\rho) = t - s(\rho)$. Fatou's lemma applied to the indicators of the strict exceedance events in (3.24) now yields

$$\mathbb{P}(|Z_\rho| > u) \le 2\exp\left(-c\min\left\{\frac{u^2}{s(\rho)}, \frac{u}{\operatorname{mesh}(\rho)}\right\}\right) \le 2\exp\left(-\frac{c\min\{u^2,u\}}{\operatorname{mesh}(\rho)}\right). \quad (3.25)$$

Fix $t \in (0,1]$ and put $\rho_n(t) = (\pi_n \cap [0,t]) \cup \{t\}$. Since $\operatorname{mesh}(\rho_n(t)) \le \operatorname{mesh}(\pi_n) = o(1/\log n)$, the last bound is summable in n for every fixed $u > 0$. Borel–Cantelli, applied with $u = 1/j$, gives $Z_{\rho_n(t)} \to 0$ almost surely. Moreover, $s(\rho_n(t)) \le t\operatorname{mesh}(\pi_n) \to 0$, and hence

$$[X]_{(\pi_n)}(t) = Z_{\rho_n(t)} + t - s(\rho_n(t)) \longrightarrow t \qquad \text{almost surely.}$$

We intersect these events over rational $t \in [0,1]$. The monotone successor sums $V_{\pi_m}(X;\cdot)$ and (2.5) then give uniform convergence on this event, by the argument at the end of the proof of Theorem 2.4.

(b) For $\ell \ge 1$, let $\mathcal{P}_\ell$ contain every partition obtained from $\mathbb{T}_{\ell-1}$ by choosing separately in each of its $2^{\ell-1}$ intervals whether to insert the midpoint. Thus

$$|\mathcal{P}_\ell| = 2^{2^{\ell-1}}, \qquad \operatorname{mesh}(\pi) \le 2^{1-\ell} \quad (\pi \in \mathcal{P}_\ell). \quad (3.26)$$

List the blocks $\mathcal{P}_1, \mathcal{P}_2, \dots$ in order, and choose a fixed ordering within each block, beginning with $\mathbb{T}_\ell$ and ending with $\mathbb{T}_{\ell-1}$. This defines a deterministic nonrefining sequence $(\widehat{\pi}_n)$. If $\widehat{\pi}_n$ occurs in block ℓ, then

$$n \le \sum_{q=1}^{\ell} 2^{2^{q-1}} < 2^{1+2^{\ell-1}}.$$

Consequently, for $n \ge 2$,

$$\operatorname{mesh}(\widehat{\pi}_n)\log n < (1 + 2^{1-\ell})\log 2 \le 2\log 2,$$

which proves the required mesh bound.

We determine the extrema within each block. Set $\lambda = 2^{-1/2}$ and define the normalized dyadic increments

$$W_{r,k} := 2^{r/2}\Big(X_{(k+1)2^{-r}} - X_{k2^{-r}}\Big), \qquad 0 \le k < 2^r.$$

The levels below r are affine on the kth level-r interval, the level-r tent has midpoint value $2^{-r/2-1}\theta_{r,k}$, and all finer levels vanish at the endpoints and midpoint. Therefore

$$W_{r+1,2k} = \lambda(W_{r,k} + \theta_{r,k}), \qquad W_{r+1,2k+1} = \lambda(W_{r,k} - \theta_{r,k}). \quad (3.27)$$

Since $W_{0,0} = 0$, induction shows that, for every function f,

$$2^{-r}\sum_{k=0}^{2^r-1} f(W_{r,k}) = \mathbb{E}f(W_r), \qquad W_r := \sum_{j=1}^{r} \lambda^j Y_j, \quad (3.28)$$

where (Y_j) is an auxiliary independent Rademacher sequence. Indeed, each pair in (3.27) contains both signs, regardless of $\theta_{r,k}$. In particular,

$$[X]_{(\mathbb{T}_\ell)} = \mathbb{E}W_\ell^2 = \sum_{j=1}^{\ell} 2^{-j} = 1 - 2^{-\ell}. \tag{3.29}$$

Fix ℓ and set $r = \ell - 1$ and $h = 2^{-\ell}$. In the kth level-r interval, the contributions obtained by inserting and by omitting the midpoint are, respectively,

$$h(W_{r,k}^2 + 1) \qquad \text{and} \qquad 2hW_{r,k}^2.$$

Thus omitting the midpoint changes the quadratic sum by $h(W_{r,k}^2 - 1)$. Every combination of these choices occurs in $\mathcal{P}_\ell$, so (3.28) gives

$$M_\ell := \max_{\pi\in\mathcal{P}_\ell}[X]_{(\pi)} = 1 - 2^{-\ell} + \frac{1}{2}\mathbb{E}[(W_{\ell-1}^2 - 1)_+], \tag{3.30}$$

$$m_\ell := \min_{\pi\in\mathcal{P}_\ell}[X]_{(\pi)} = 1 - 2^{-\ell} - \frac{1}{2}\mathbb{E}[(1 - W_{\ell-1}^2)_+]. \tag{3.31}$$

Both extrema are deterministic, although the partitions attaining them may depend on the signs.

The variables W_r converge uniformly in the auxiliary signs to

$$W := \sum_{j=1}^{\infty} 2^{-j/2} Y_j = \sqrt{2}\,U + V, \qquad U := \sum_{j=1}^{\infty} 2^{-j} Y_{2j-1}, \quad V := \sum_{j=1}^{\infty} 2^{-j} Y_{2j}.$$

The variables U and V are independent and uniform on $[-1,1]$. Writing $b = \sqrt{2}$, the convolution density of W is $f_W(x) = (1 + b - |x|)/(4b)$ for $1 \le |x| \le 1 + b$. Hence

$$\mathbb{E}[(W^2 - 1)_+] = \frac{1}{2b}\int_1^{1+b} (x^2 - 1)(1 + b - x)\,dx = \frac{4 + \sqrt{2}}{12} = 2\gamma. \tag{3.32}$$

Since $\mathbb{E}W^2 = 1$, we also have $\mathbb{E}[(1 - W^2)_+] = 2\gamma$. Bounded convergence in (3.30)–(3.31) yields $M_\ell \to 1 + \gamma$ and $m_\ell \to 1 - \gamma$. Every quadratic sum in block ℓ lies between these extrema, and both extrema are attained. It follows that

$$\liminf_{n\to\infty}[X]_{(\widehat{\pi}_n)} = 1 - \gamma, \qquad \limsup_{n\to\infty}[X]_{(\widehat{\pi}_n)} = 1 + \gamma.$$

The construction and all identities in part (b) hold for every sign array. □

Proof of Theorem 2.6. The identity (3.29) holds for every sign array, so it also proves (2.12).

To separate the effect of the bias, define the periodic tent function and the all-positive path by

$$\varphi(t) := \operatorname{dist}(t, \mathbb{Z}), \qquad F(t) := \sum_{m=0}^{\infty} 2^{-m/2}\varphi(2^m t).$$

Set $\zeta_{m,k}^{(\nu)} := \theta_{m,k}^{(\nu)} - b$ and $\sigma^2 := \mathbb{E}(\zeta_{m,k}^{(\nu)})^2 = 1 - b^2$. Then

$$Y^{(\nu)} = bF + \widetilde{Y}, \qquad \widetilde{Y}_t := \sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1} \zeta_{m,k}^{(\nu)} e_{m,k}(t), \tag{3.33}$$

where the coefficients of $\widetilde{Y}$ are independent, centered, and bounded. We abbreviate these coefficients to $\zeta_\alpha^{(\nu)}$ when using the single Haar index α.

For integers $q \geq 1$, let $\tau_q := \{j/q : 0 \leq j \leq q\}$ and write $\Delta_j f := f((j+1)/q) - f(j/q)$ in calculations on this grid. The operators from Lemma 3.4 satisfy

$$\operatorname{tr}(A_{\tau_q}) = 1 - q^{-1}, \qquad \|A_{\tau_q}\|_{\mathrm{HS}}^2 \leq \|K_{\tau_q}\|_{\mathrm{HS}}^2 = q^{-1}.$$

Expanding the quadratic form in the Haar basis gives

$$\mathbb{E}\Big([\widetilde{Y}]_{(\tau_q)}\Big) = \sigma^2(1 - q^{-1}), \qquad \operatorname{Var}\Big([\widetilde{Y}]_{(\tau_q)}\Big) \leq C_\nu q^{-1}. \tag{3.34}$$

Indeed, the variance is

$$\operatorname{Var}((\zeta_{0,0}^{(\nu)})^2) \sum_\alpha (A_{\tau_q})_{\alpha\alpha}^2 + 2\sigma^4 \sum_{\alpha \neq \beta} (A_{\tau_q})_{\alpha\beta}^2.$$

These identities follow first for finite truncations; boundedness of the coefficients and convergence of the compressed operators in trace and Hilbert–Schmidt norms justify passage to the limit.

For the cross term, set

$$C_q := \sum_{j=0}^{q-1} \Delta_j F \, \Delta_j \widetilde{Y}, \qquad g_q := \sum_{j=0}^{q-1} (\Delta_j F) \mathbf{1}_{(j/q,(j+1)/q]}.$$

Since $F(0) = F(1) = 0$, we have $Pg_q = g_q$. Parseval's identity and Lemma 3.3 yield

$$\mathbb{E}C_q = 0, \qquad \operatorname{Var}(C_q) = \sigma^2 \|g_q\|_2^2 = \frac{\sigma^2}{q}[F]_{(\tau_q)} \leq C'_\nu q^{-1}. \tag{3.35}$$

The bounds in (3.34) and (3.35) are summable along both $q = 3 \cdot 2^n$ and $q = 3^n$. Chebyshev's inequality and the Borel–Cantelli lemma therefore imply, on one event of probability one,

$$[Y^{(\nu)}]_{(\tau_q)} = \sigma^2 + b^2[F]_{(\tau_q)} + o(1) \tag{3.36}$$

along both sequences. It remains to compute the deterministic sums.

For the thirds grids, put

$$a := F(1/3) = F(2/3) = \frac{1}{3} \sum_{m=0}^{\infty} 2^{-m/2} = \frac{2 + \sqrt{2}}{3}.$$

On each level-n dyadic cell, the levels below n are affine and the remaining levels form a copy of $2^{-n/2}F$. If $d_{n,k} := F((k+1)2^{-n}) - F(k2^{-n})$, the three increments within that cell are consequently

$$\frac{d_{n,k}}{3} + 2^{-n/2}a, \qquad \frac{d_{n,k}}{3}, \qquad \frac{d_{n,k}}{3} - 2^{-n/2}a.$$

Squaring and summing gives the exact identity

$$[F]_{(\mathbb{U}_n)} = \frac{1}{3}[F]_{(\mathbb{T}_n)} + 2a^2 = \frac{1}{3}(1 - 2^{-n}) + \frac{2(2 + \sqrt{2})^2}{9}. \tag{3.37}$$

Its limit is $(15+8\sqrt{2})/9$. Since $\sigma^2 = 1-b^2$, (3.36) proves (2.9).

To treat the triadic grids, we establish an asymptotic formula for every odd q. Write

$$L := \lfloor \log_2 q \rfloor, \qquad c_q := q2^{-L} \in [1,2),$$

and define

$$\begin{aligned}
\mathcal{R}(u) &:= \int_0^1 \Big(\varphi(x+u)-\varphi(x)\Big)^2\,dx \\
&= d(u)^2 - \frac{4}{3}d(u)^3, \qquad d(u) := \operatorname{dist}(u,\mathbb{Z}), && (3.38)\\
G(c) &:= c\sum_{\ell\in\mathbb{Z}} 2^{-\ell}\mathcal{R}(2^\ell/c), \qquad 1\le c\le 2. && (3.39)
\end{aligned}$$

The formula for $\mathcal{R}$ follows by integration on the linear pieces of φ. The summands defining G are bounded uniformly in c by constant multiples of $2^{-\ell}$ for $\ell \ge 0$ and 2^ℓ for $\ell < 0$. Thus the series converges uniformly and G is continuous. We claim that

$$[F]_{(\tau_q)} - G(c_q) \longrightarrow 0 \qquad \text{as odd } q \longrightarrow \infty. \tag{3.40}$$

The first step is to compare the discrete sum with its spatial average. Using the periodic extension of F, set

$$D_q(t) := \sqrt{q}\Big(F(t+q^{-1}) - F(t)\Big).$$

The contribution of level m is bounded by

$$\min\left\{\frac{2^{m/2}}{\sqrt{q}}, \frac{\sqrt{q}}{2}\,2^{-m/2}\right\}.$$

Consequently, $\|D_q\|_\infty$ is bounded uniformly in q. For a fixed integer $R \ge 0$ and $L \ge R$, let $D_q^{[R]}$ retain only the levels $L-R,\dots,L+R$. Summing the omitted geometric tails gives $\|D_q - D_q^{[R]}\|_\infty = O(2^{-R/2})$, uniformly in q. Define

$$H_{R,c}(x) := \sum_{\ell=-R}^{R} \sqrt{c}\,2^{-\ell/2}\left[\varphi(2^{\ell+R}x + 2^\ell/c) - \varphi(2^{\ell+R}x)\right].$$

Then $D_q^{[R]}(t) = H_{R,c_q}(2^{L-R}t)$. Because q is odd, multiplication by 2^{L-R} permutes the residues modulo q. Periodicity therefore gives

$$\begin{aligned}
\frac{1}{q}\sum_{j=0}^{q-1}\Big|D_q^{[R]}(j/q)\Big|^2 &= \frac{1}{q}\sum_{j=0}^{q-1}\Big|H_{R,c_q}(j/q)\Big|^2,\\
\int_0^1 \Big|D_q^{[R]}(t)\Big|^2\,dt &= \int_0^1 \Big|H_{R,c_q}(x)\Big|^2\,dx.
\end{aligned}$$

For fixed R, the functions $H_{R,c}^2$, $1\le c\le 2$, have a common Lipschitz bound, so the difference between the last Riemann sum and integral is $O_R(q^{-1})$. Restoring the omitted levels gives an error of $O(2^{-R/2}) + O_R(q^{-1})$. Letting $q\to\infty$ through odd integers and then $R\to\infty$ proves

$$[F]_{(\tau_q)} - q\int_0^1 \Big(F(t+q^{-1}) - F(t)\Big)^2\,dt \longrightarrow 0. \tag{3.41}$$

The level increments in this integral are orthogonal. Indeed, $\varphi(x+1/2) = 1/2-\varphi(x)$, so translation by 2^{-m-1} changes the sign of the level-m increment and leaves every finer-level increment unchanged. Their pairwise integrals therefore vanish, and

$$q\int_0^1 \Big(F(t+q^{-1})-F(t)\Big)^2\,dt = q\sum_{m=0}^{\infty} 2^{-m}\mathcal{R}(2^m/q)$$
$$= c_q \sum_{\ell=-L}^{\infty} 2^{-\ell}\mathcal{R}(2^\ell/c_q).$$

The omitted terms with $\ell < -L$ are $O(2^{-L})$, uniformly in c_q. Together with (3.41), this proves (3.40).

The negative-index terms in (3.39) sum to $c^{-1}-4/(9c^2)$. For $c=1$, all remaining terms vanish; for $c=3/2$, they have $d(2^\ell/c)=1/3$ and $\mathcal{R}(2^\ell/c)=5/81$. Hence

$$G(1)=\frac{5}{9}, \qquad G(3/2)=\frac{2}{3}-\frac{16}{81}+\frac{15}{81}=\frac{53}{81}. \tag{3.42}$$

Finally,

$$c_{3^n} = 2^{\{n\log_2 3\}}.$$

Since $\log_2 3$ is irrational, the fractional parts $\{n\log_2 3\}$ are dense in $[0,1]$. We may therefore choose deterministic subsequences (n_j) and (m_j) such that $c_{3^{n_j}}\to 1$ and $c_{3^{m_j}}\to 3/2$. Continuity of G, (3.36), and (3.40) now give, almost surely,

$$[Y^{(\nu)}]_{(\mathbb{T}^{(3)}_{n_j})} \longrightarrow \sigma^2+\frac{5}{9}b^2 = 1-\frac{4}{9}b^2,$$
$$[Y^{(\nu)}]_{(\mathbb{T}^{(3)}_{m_j})} \longrightarrow \sigma^2+\frac{53}{81}b^2 = 1-\frac{28}{81}b^2.$$

These are (2.10) and (2.11), and they are distinct because $b\neq 0$. □

3.3 Critical variation

We now prove Theorem 2.7 and Proposition 2.8. We first establish a common limit formula for partitions obtained by repeating a fixed template in each dyadic cell. A small dyadic split then gives partition dependence for every $H\neq 1/2$, while the thirds template permits an explicit calculation at $H=1/4$.

3.3.1 Finite-template limits

Fix $H\in(0,1)$, set $p=1/H$ and $\lambda=2^{H-1}$, and let

$$W_n := \sum_{r=1}^{n}\lambda^r Y_{r-1}, \qquad W := \lambda Z_H = \sum_{r=1}^{\infty}\lambda^r Y_{r-1},$$

where the auxiliary Rademacher sequence (Y_m) from (2.6) is independent of X^H. For a finite template $\boldsymbol{u}=\{0=u_0<u_1<\cdots<u_q=1\}$, define

$$\mathbb{P}_n(\boldsymbol{u}) := \Big\{2^{-n}(k+u_j) : 0\le k<2^n,\ 0\le j\le q\Big\}, \tag{3.43}$$

with repeated endpoints included only once, and set

$$G_{\boldsymbol{u},p}(w,x) := \sum_{j=0}^{q-1} |(u_{j+1}-u_j)w + x(u_{j+1}) - x(u_j)|^p\,. \tag{3.44}$$

The sequence $\big(\mathbb{P}_n(\boldsymbol{u})\big)$ need not be refining; the following limit requires only that the template be fixed.

Lemma 3.5 (Finite-template limits)**.** *For every sign array,*

$$[X^H]^{(p)}_{(\mathbb{T}_n)} = \mathbb{E}|W_n|^p \longrightarrow \mathbb{E}|W|^p = C_{\mathrm{dyad}}(H). \tag{3.45}$$

For every fixed finite template $\boldsymbol{u}$,

$$[X^H]^{(p)}_{(\mathbb{P}_n(\boldsymbol{u}))} \longrightarrow C_{\boldsymbol{u}}(H) := \mathbb{E}G_{\boldsymbol{u},p}(W,X^H) \qquad \textit{almost surely.} \tag{3.46}$$

Proof. On the dyadic cell $I_{n,k} = [k2^{-n},(k+1)2^{-n}]$, the levels $m<n$ are affine. The level-m term has slope of magnitude $2^{m(1-H)}$. Writing $r=n-m$, its slope after rescaling the cell and dividing by 2^{-nH} therefore has magnitude $2^{nH-n}2^{(n-r)(1-H)} = \lambda^r$. Consequently,

$$X^H_{2^{-n}(k+u)} - X^H_{2^{-n}k} = 2^{-nH}\Big(uW_{n,k} + \widetilde{X}^{n,k}_u\Big), \qquad 0\le u\le 1, \tag{3.47}$$

where, for suitable slope signs $\sigma^{(n)}_{r,k}\in\{-1,1\}$,

$$W_{n,k} := \sum_{r=1}^{n} \lambda^r \sigma^{(n)}_{r,k}, \tag{3.48}$$

and the descendant terms form the process

$$\widetilde{X}^{n,k}_u := \sum_{\ell=0}^{\infty} 2^{\ell(1/2-H)} \sum_{v=0}^{2^\ell-1} \theta_{n+\ell,\,2^\ell k+v}\, e_{\ell,v}(u).$$

Let $\mathcal{F}_{<n}$ be generated by the signs at levels below n. The processes $\widetilde{X}^{n,k}$ use disjoint coefficient subtrees; conditionally on $\mathcal{F}_{<n}$, they are independent copies of X^H.

For every sign array, the slope vectors $(\sigma^{(n)}_{1,k},\dots,\sigma^{(n)}_{n,k})$, $0\le k<2^n$, enumerate $\{-1,1\}^n$ exactly once. Indeed, bisecting a cell preserves all its previous slope signs and adds the two opposite signs $+\theta_{n,k}$ and $-\theta_{n,k}$. Induction, with the coordinate order reversed as in (3.48), proves the assertion. Since $\widetilde{X}^{n,k}_1 = 0$ and $Hp=1$, it follows that

$$[X^H]^{(p)}_{(\mathbb{T}_n)} = 2^{-n}\sum_{k=0}^{2^n-1} |W_{n,k}|^p = \mathbb{E}|W_n|^p.$$

The variables W_n converge to W and are uniformly bounded. Thus bounded convergence and $\lambda^p = 2^{1-p}$ give (3.45), with the normalization in (2.6).

For the general template, (3.47) gives

$$[X^H]^{(p)}_{(\mathbb{P}_n(\boldsymbol{u}))} = 2^{-n}\sum_{k=0}^{2^n-1} G_{\boldsymbol{u},p}(W_{n,k},\widetilde{X}^{n,k}). \tag{3.49}$$

Set $g(w) := \mathbb{E}G_{\boldsymbol{u},p}(w, X^H)$. Conditional independence and the slope-vector enumeration imply

$$\mathbb{E}\left[[X^H]^{(p)}_{(\mathbb{P}_n(\boldsymbol{u}))} \mid \mathcal{F}_{<n}\right] = 2^{-n}\sum_{k=0}^{2^n-1} g(W_{n,k}) = \mathbb{E}g(W_n) =: m_n. \tag{3.50}$$

Uniformly over the signs,

$$|W_{n,k}| \le \frac{\lambda}{1-\lambda}, \qquad \|X^H\|_\infty \le \frac{1}{2(1-2^{-H})}.$$

Hence g is continuous and bounded on the common range of W_n and W, so $m_n \to \mathbb{E}g(W)$. The same bounds give a deterministic constant B bounding every summand in (3.49). Conditional independence therefore yields

$$\mathbb{E}\left|[X^H]^{(p)}_{(\mathbb{P}_n(\boldsymbol{u}))} - m_n\right|^2 \le B^2 2^{-n}.$$

By Tonelli's theorem, the squared centered terms have a finite sum almost surely and thus tend to zero. Together with the limit of m_n, this proves (3.46). □

3.3.2 The small-split comparison

Proof of Theorem 2.7. Retain p, λ, and W from Lemma 3.5, and write $M_p := \mathbb{E}|W|^p = C_{\mathrm{dyad}}(H)$. For $r \ge 1$, let $t_r = 2^{-r}$ and $\boldsymbol{u}_r = \{0, t_r, 1\}$, so that $\mathbb{P}_n(\boldsymbol{u}_r) = \mathbb{V}_n^{(r)}$. Lemma 3.5 gives

$$\begin{aligned}[X^H]^{(p)}_{(\mathbb{V}_n^{(r)})} &\longrightarrow C_r(H) \qquad \text{almost surely},\\ C_r(H) &:= \mathbb{E}\left[|t_r W + X^H_{t_r}|^p + |(1-t_r)W - X^H_{t_r}|^p\right].\end{aligned} \tag{3.51}$$

We compare $C_r(H)$ with M_p as $r \to \infty$. Write $t = t_r$ and $a = t^H$. At the dyadic point $t = 2^{-r}$, all levels at least r vanish and the remaining tents are on their increasing branches. Thus

$$X^H_t = t\sum_{m=0}^{r-1} 2^{m(1-H)}\theta_{m,0} \overset{d}{=} aU_r, \qquad U_r := \sum_{j=1}^{r}\lambda^j\xi_j, \tag{3.52}$$

where (ξ_j) is an independent Rademacher sequence, independent of W. Since $a\lambda^r = t$ and $U_r + \lambda^r W \overset{d}{=} W$, the first term in (3.51) contributes $a^p M_p = tM_p$. Symmetry of U_r then gives

$$C_r(H) = tM_p + \mathbb{E}|(1-t)W + aU_r|^p. \tag{3.53}$$

With $c_r = a/(1-t)$, we obtain the comparison identity

$$C_r(H) - M_p = \left[t + (1-t)^p - 1\right]M_p + (1-t)^p\left(\mathbb{E}|W + c_rU_r|^p - M_p\right). \tag{3.54}$$

Here W and all U_r are bounded by $\lambda/(1-\lambda)$, $c_r \to 0$, and $\mathbb{E}U_r^2 \to \mathbb{E}W^2 > 0$.

Suppose first that $H < 1/2$, so $p > 2$. The function $f(x) = |x|^p$ has a continuous second derivative. Taylor's formula, uniformly on the common bounded range, and $\mathbb{E}U_r = 0$ therefore yield

$$\mathbb{E}|W + c_rU_r|^p = M_p + \frac{p(p-1)}{2}c_r^2\mathbb{E}|W|^{p-2}\,\mathbb{E}U_r^2 + o(c_r^2). \tag{3.55}$$

Since $t + (1-t)^p - 1 = (1-p)t + O(t^2)$, $(1-t)^p c_r^2 = t^{2H}(1-t)^{p-2}$, and $t = o(t^{2H})$, (3.54) gives

$$C_r(H) - M_p \sim \frac{p(p-1)}{2}\, t^{2H}\mathbb{E}|W|^{p-2}\,\mathbb{E}W^2 > 0. \tag{3.56}$$

Now suppose that $H > 1/2$, so $1 < p < 2$. We first note that W has no atoms. Indeed, its law μ satisfies $W \overset{d}{=} \lambda(Y + W')$, where Y is Rademacher and W' is an independent copy of W. If μ had atoms, its largest atom mass $m > 0$ would be attained, and the set $A = \{x : \mu(\{x\}) = m\}$ would be finite and nonempty. For every $x \in A$,

$$m = \frac{1}{2}\mu\left(\{x/\lambda - 1\}\right) + \frac{1}{2}\mu\left(\{x/\lambda + 1\}\right).$$

Both masses on the right must equal m. Thus, with $\alpha = \min A$ and $\beta = \max A$, the points $\alpha/\lambda - 1$ and $\beta/\lambda + 1$ belong to A but are separated by $(\beta - \alpha)/\lambda + 2 > \beta - \alpha$, a contradiction.

The derivative of $f(x) = |x|^p$ is $(p-1)$-Hölder continuous, so

$$\Big||x+y|^p - |x|^p - p|x|^{p-2}xy\Big| \le K_p|y|^p, \qquad x, y \in \mathbb{R}, \tag{3.57}$$

where $|x|^{p-2}x$ is interpreted as zero at $x = 0$. For $W \neq 0$, the ordinary second-order Taylor expansion and the uniform bound on U_r imply

$$\frac{|W + c_rU_r|^p - |W|^p - p|W|^{p-2}W c_rU_r}{c_r^p} = O_W(c_r^{2-p}) \longrightarrow 0.$$

The bound (3.57) dominates these quotients by $K_p(\lambda/(1-\lambda))^p$. Since $\mathbb{P}(W = 0) = 0$, dominated convergence, independence, and $\mathbb{E}U_r = 0$ give

$$\mathbb{E}|W + c_rU_r|^p = M_p + o(c_r^p). \tag{3.58}$$

As $(1-t)^p c_r^p = t$, (3.54) now yields

$$C_r(H) - M_p = (1-p)tM_p + o(t) < 0 \tag{3.59}$$

for all sufficiently large r.

In either case, fix r large enough that $C_r(H) \neq M_p$ with the claimed sign. The split points are dyadic of level $n + r$, so

$$\mathbb{T}_n \subseteq \mathbb{V}_n^{(r)} \subseteq \mathbb{T}_{n+r}. \tag{3.60}$$

It follows that the sequence in (2.14) is refining and has vanishing mesh. By (3.51) and (3.45), its odd terminal sums converge almost surely to $C_r(H)$ and its even terminal sums to M_p. Their distinct limits prove the theorem. □

At $H = 1/2$, the same calculation gives $p = 2$, $M_2 = 1$, and $\mathbb{E}U_r^2 = 1 - t_r$. Hence (3.53) yields $C_r(1/2) = 1$ for every r, in agreement with Theorem 2.4.

3.3.3 The explicit thirds-grid limit

Proof of Proposition 2.8. Take the template $\boldsymbol{u}_\star = \{0, 1/3, 2/3, 1\}$. Then $\mathbb{P}_n(\boldsymbol{u}_\star) = \mathbb{U}_n$. Lemma 3.5 therefore gives

$$C_{\text{thirds}} = \mathbb{E}\sum_{j=0}^{2}\left|\frac{1}{3}W + X^{1/4}_{(j+1)/3} - X^{1/4}_{j/3}\right|^4, \tag{3.61}$$

where $\lambda = 2^{-3/4}$ and W is independent of $X^{1/4}$. Write $b = 2^{-1/4}$. The dyadic orbit of $1/3$ or $2/3$ is always at relative position $1/3$ or $2/3$ in its tent. Consequently,

$$X^{1/4}_{1/3} = \frac{1}{3}\sum_{m\geq 0} b^m \xi_m, \qquad X^{1/4}_{2/3} = \frac{1}{3}\sum_{m\geq 0} b^m \zeta_m.$$

The two points share the level-zero tent, so $\xi_0 = \zeta_0$. At every higher level they lie in distinct tents. Thus ξ_0 and $\{\xi_m, \zeta_m : m \geq 1\}$ are independent Rademacher variables, also independent of W.

Denote the three increments in (3.61) by D_1, D_2, D_3. Then

$$3D_1 = W + \sum_{m\geq 0} b^m \xi_m, \qquad 3D_2 = W + \sum_{m\geq 1} b^m(\zeta_m - \xi_m),$$

and $D_3 \overset{d}{=} D_1$. For an absolutely summable coefficient sequence and independent Rademacher variables (η_i),

$$\mathbb{E}\left(\sum_i a_i\eta_i\right)^4 = 3\left(\sum_i a_i^2\right)^2 - 2\sum_i a_i^4. \tag{3.62}$$

This follows by expanding finite sums and then using bounded convergence. The coefficient sums for the three series needed here are geometric:

series	$\sum_i a_i^2$	$\sum_i a_i^4$
W	$\frac{1+2\sqrt{2}}{7}$	$\frac{1}{7}$
$3D_1$	$\frac{15+9\sqrt{2}}{7}$	$\frac{15}{7}$
$3D_2$	$\frac{15+16\sqrt{2}}{7}$	$\frac{15}{7}$

Applying (3.62) gives

$$C_{\text{dyad}}(1/4) = \mathbb{E}W^4 = \frac{13+12\sqrt{2}}{49}, \tag{3.63}$$

and

$$\mathbb{E}D_1^4 = \frac{317+270\sqrt{2}}{1323}, \qquad \mathbb{E}D_2^4 = \frac{667+480\sqrt{2}}{1323}.$$

Hence

$$C_{\text{thirds}} = 2\mathbb{E}D_1^4 + \mathbb{E}D_2^4 = \frac{1301+1020\sqrt{2}}{1323},$$

and subtraction gives the positive difference in (2.16). $\square$

3.4 Invariance of variation index

3.4.1 Proofs

Throughout this subsection $H \in (0,1)$, $p = 1/H$, and we write

$$f_{m,k} := 2^{m(1/2-H)} e_{m,k}, \quad \|f_{m,k}\|_\infty = 2^{-mH-1},$$
$$\mathrm{Lip}(f_{m,k}) = 2^{m(1-H)}, \qquad \operatorname{supp} f_{m,k} = [k2^{-m}, (k+1)2^{-m}],$$

so that $X^H = \sum_{m,k} \theta_{m,k} f_{m,k}$. For an interval $I = (u,v]$ and a function g we write $\Delta_I g := g(v) - g(u)$ and $|I| := v - u$.

Proof of Proposition 2.9. By Lemma 3.3 and $pH = 1$,

$$[X^H]^{(p)}_{(\tau)} = \sum_{I\in\tau} |\Delta_I X^H|^p \le L_H^p \sum_{I\in\tau} |I|^{pH} = L_H^p \sum_{I\in\tau} |I| = L_H^p.$$

For $q > p$, $[X^H]^{(q)}_{(\pi_n)} \le \omega_{X^H}(\operatorname{mesh} \pi_n)^{q-p} [X^H]^{(p)}_{(\pi_n)} \le L_H^q \operatorname{mesh}(\pi_n)^{(q-p)H} \to 0$. Hence $p^\pi(X^H) \le p$ by Definition 2.3. □

Lemma 3.6 (Second moment of an increment)**.** *There is a constant $c_H > 0$ such that, for every interval $I = (u,v]$ with $0 < |I| \le 1/2$,*

$$c_H |I|^{2H} \le \mathbb{E}\Big(\Delta_I X^H\Big)^2 \le L_H^2 |I|^{2H}. \tag{3.64}$$

Consequently there is $c'_H > 0$ such that $\mathbb{E}|\Delta_I X^H|^p \ge c'_H |I|$ for $0 < |I| \le 1/2$, and

$$\mathbb{E}\,[X^H]^{(p)}_{(\tau)} \ge c'_H \qquad \textit{for every partition } \tau \textit{ with } \operatorname{mesh}(\tau) \le 1/2. \tag{3.65}$$

Proof. The upper bound is Lemma 3.3. For the lower bound, let $h_{m,k} = e'_{m,k}$ be the Haar functions of the proof of Lemma 3.4. Since $\Delta_I f_{m,k} = 2^{m(1/2-H)} \int_u v h_{m,k}$ and the signs are independent with unit variance,

$$\mathbb{E}\Big(\Delta_I X^H\Big)^2 = \sum_{m\ge 0} 2^{m(1-2H)} \sum_{k<2^m} \Big(\int_u^v h_{m,k}\Big)^2. \tag{3.66}$$

Write $\delta = |I|$ and choose $N \ge 0$ with $2^{-N-1} < \delta \le 2^{-N}$.

Suppose first that $\delta \le 1/16$, so $N \ge 4$. The dyadic points of level at most $N-2$ are spaced $2^{-N+2} \ge 4\delta$ apart, so I contains at most one of them; call it d if it exists, and let $j \le N-2$ be its level. Fix a level $m \le N-3$. The endpoints and midpoints of the level-m cells are dyadic points of level at most $N-2$. If none of them lies in I, then I is contained in a single level-m cell on one side of its midpoint, exactly one k contributes to the inner sum in (3.66), and its contribution is $(2^{m/2}\delta)^2 = 2^m \delta^2$. If $d \in I$ is an endpoint of a level-m cell, then I meets two cells, each piece lies on one side of the midpoint of its cell because its length is less than a quarter of the cell, and the inner sum equals $2^m\Big((d-u)^2 + (v-d)^2\Big) \ge 2^m\delta^2/2$. If d is the midpoint of the level-m cell containing I, which happens only for $m = j-1$, the contribution may be smaller and we discard it. Hence

$$\mathbb{E}\Big(\Delta_I X^H\Big)^2 \ge \frac{\delta^2}{2}\Big(\sum_{m=0}^{N-3} 2^{m(2-2H)} - 2^{(N-3)(2-2H)}\Big) \ge \frac{\delta^2}{2}\, 2^{(N-4)(2-2H)},$$

where we used that the sum is at least $2^{(N-3)(2-2H)}(1+2^{-(2-2H)})$. Since $2^N > (2\delta)^{-1}$, this is at least $c_H\delta^{2H}$ with $c_H = 2^{-1-6(2-2H)}$.

For $1/16 < \delta \le 1/2$, the function $(u,v) \mapsto \mathbb{E}(\Delta_I X^H)^2$ is continuous, because the series (2.1) converges uniformly, and it is strictly positive: it vanishes only if $\int_u^v h_{m,k} = 0$ for all (m,k), i.e. if $\mathbf{1}_{(u,v]}$ is orthogonal to H_0 and hence constant a.e., which forces $\delta \in \{0,1\}$. A continuous positive function on the compact set $\{(u,v) : 1/16 \le v-u \le 1/2\}$ is bounded below by a positive constant, and (3.64) follows after decreasing c_H. The cutoff $1/2$ can be replaced by any fixed $0 < a < 1$, with the lower-bound constant also depending on a. A cutoff away from one is necessary: for $I = (0, 1-\varepsilon]$, the bridge property gives $\mathbb{E}(\Delta_I X^H)^2 \le L_H^2\varepsilon^{2H} \to 0$, whereas $|I|^{2H} \to 1$.

For the moment bound, write $Z = \Delta_I X^H$, so $|Z| \le L_H\delta^H$. If $p \ge 2$, Jensen's inequality gives $\mathbb{E}|Z|^p \ge (\mathbb{E}Z^2)^{p/2} \ge c_H^{p/2}\delta^{pH} = c_H^{p/2}\delta$. If $1 < p < 2$, then $Z^2 = |Z|^p|Z|^{2-p} \le |Z|^p(L_H\delta^H)^{2-p}$, so $\mathbb{E}|Z|^p \ge \mathbb{E}Z^2\,(L_H\delta^H)^{p-2} \ge c_H L_H^{p-2}\delta^{pH} = c_H L_H^{p-2}\delta$. Summing over the intervals of τ gives (3.65). □

Lemma 3.7 (Bounded differences of the critical sum)**.** *Let τ be any partition of $[0,1]$ with $h := \mathrm{mesh}(\tau) \le 1/2$. Let $S(\theta) := [X^H]^{(p)}_{(\tau)}$, viewed as a function of the sign array $\theta = (\theta_{m,k})$, and let $c_{m,k}$ denote the maximal change of S when $\theta_{m,k}$ alone is flipped. Then*

$$\sum_{m\ge 0}\sum_{k<2^m} c_{m,k}^2 \le C_H\Big(h^{\kappa_H} + h\mathbf{1}_{\{H=1/2\}}\log(e/h)\Big), \tag{3.67}$$

with κ_H as in (2.17) *and a constant C_H depending only on H.*

Proof. Flipping $\theta_{m,k}$ replaces X^H by $X^H \mp 2f_{m,k}$, which is again a member of the class (2.1). By the mean value theorem, $\big||a|^p - |b|^p\big| \le p\max(|a|,|b|)^{p-1}|a-b|$, and both a and b are increments over I of functions in the class, so Lemma 3.3 and $(p-1)H = 1-H$ give

$$c_{m,k} \le 2pL_H^{p-1}\sum_{I\in\tau}|I|^{1-H}\,|\Delta_I f_{m,k}|. \tag{3.68}$$

Put $\delta_I := |I|$ and $\ell_m := 2^{-m}$. The total variation of $f_{m,k}$ is ℓ_m^H, twice its maximum height. Hence $\sum_{I\in\tau}|\Delta_I f_{m,k}| \le \ell_m^H$, and weighted Cauchy–Schwarz applied to (3.68) gives

$$c_{m,k}^2 \le C_H\ell_m^H\sum_{I\in\tau}\delta_I^{2-2H}|\Delta_I f_{m,k}|. \tag{3.69}$$

At each level, the tent derivatives satisfy $\sum_{k<2^m}|f'_{m,k}(t)| = \ell_m^{H-1}$ for almost every t. Integration over I therefore bounds $\sum_k|\Delta_I f_{m,k}|$ by $\ell_m^{H-1}\delta_I$. Also, each endpoint belongs to at most one nonzero tent, whose height is at most $\ell_m^H/2$. The triangle inequality gives the second bound in

$$\sum_{k<2^m}|\Delta_I f_{m,k}| \le \min\{\ell_m^{H-1}\delta_I, \ell_m^H\}. \tag{3.70}$$

Combining these estimates and summing nonnegative terms yields

$$\sum_{m\ge 0}\sum_{k<2^m} c_{m,k}^2 \le C_H\sum_{I\in\tau}\delta_I^{2-2H}Q_H(\delta_I),$$

where

$$Q_H(\delta) := \delta \sum_{\ell_m \geq \delta} \ell_m^{2H-1} + \sum_{\ell_m < \delta} \ell_m^{2H}.$$

The two geometric sums imply

$$Q_H(\delta) \leq C_H \begin{cases} \delta^{2H}, & H < 1/2, \\ \delta \log(e/\delta), & H = 1/2, \\ \delta, & H > 1/2. \end{cases}$$

Consequently,

$$\sum_{m \geq 0} \sum_{k < 2^m} c_{m,k}^2 \leq C_H \begin{cases} h, & H < 1/2, \\ h \log(e/h), & H = 1/2, \\ h^{2-2H}, & H > 1/2. \end{cases}$$

Here we used $\sum_I \delta_I = 1$ and $\delta_I \leq h$; in the middle case, $t \mapsto t \log(e/t)$ is increasing on $(0, 1]$. This proves (3.67) without any restriction on the relative lengths of the partition intervals. □

Proof of Theorem 2.10. Write $S_n := [X^H]^{(p)}_{(\pi_n)}$ and $h_n := \mathrm{mesh}(\pi_n)$. For n large, $h_n \leq 1/2$, and (2.20) follows from Proposition 2.9 and Lemma 3.6.

For the concentration, let $S_n^{[M]}$ be the same sum computed from the truncation $X^{[M]} = \sum_{m<M} \sum_k \theta_{m,k} f_{m,k}$. It is a function of finitely many independent signs. The proof of Lemma 3.3 gives the same Hölder constant L_H when levels are omitted, both before and after a sign flip. Thus the explicit sensitivity estimates in (3.68)–(3.70) apply to $S_n^{[M]}$ as well. Summing them as in Lemma 3.7 bounds the sum of squared sensitivities by $C_H h_n^{\kappa_H}$, uniformly in M, when $H \neq 1/2$. McDiarmid's inequality therefore gives, for every $\varepsilon > 0$,

$$\mathbb{P}\Big(|S_n^{[M]} - \mathbb{E} S_n^{[M]}| > \varepsilon\Big) \leq 2 \exp\Big(-\frac{2\varepsilon^2}{C_H} h_n^{\kappa_H}\Big).$$

As $M \to \infty$, $X^{[M]} \to X^H$ uniformly, hence $S_n^{[M]} \to S_n$ for every sign array and, by bounded convergence, $\mathbb{E} S_n^{[M]} \to \mathbb{E} S_n$. Fatou's lemma applied to the indicators of the strict exceedance events yields

$$\mathbb{P}\Big(|S_n - \mathbb{E} S_n| > \varepsilon\Big) \leq 2 \exp\Big(-\frac{2\varepsilon^2}{C_H} h_n^{\kappa_H}\Big). \tag{3.71}$$

Under (2.19), $h_n^{\kappa_H} \log n \to 0$, so the right-hand side is $2n^{-a_n}$ with $a_n \to \infty$, which is summable. Borel–Cantelli with $\varepsilon = 1/j$, intersected over $j \in \mathbb{N}$, gives (2.21). Combining it with (2.20) gives (2.22).

Finally, $p^\pi(X^H) \leq p$ by Proposition 2.9. For $1 \leq q < p$, on the event (2.22),

$$[X^H]^{(q)}_{(\pi_n)} \geq \frac{[X^H]^{(p)}_{(\pi_n)}}{\omega_{X^H}(h_n)^{p-q}} \geq \frac{c_H/2}{L_H^{p-q} h_n^{(p-q)H}} \longrightarrow \infty$$

for all large n, so $p^\pi(X^H) \geq p$ by (2.8). □

Proof of Corollary 2.11. (i)⇒(ii) is trivial. If (ii) holds, then S_n converges in probability and, by (2.21), $\mathbb{E}S_n = S_n - (S_n - \mathbb{E}S_n)$ converges in probability; a deterministic sequence converging in probability converges, which is (iii). If (iii) holds, (2.21) gives $S_n \to \lim_n \mathbb{E}S_n$ almost surely, which is (i) together with the identification of the limit. □

Remark 5. Lemma 3.7 applies to arbitrary finite partitions, so Theorem 2.10 requires neither balance nor a bound on the number of partition intervals. At $H = 1/2$, the same bounded-differences argument gives almost-sure convergence of the centered quadratic sums to zero under

$$\operatorname{mesh}(\pi_n) \log(e/\operatorname{mesh}(\pi_n)) = o(1/\log n).$$

This sufficient condition is stronger than the one in Theorem 2.5(a). The Hanson–Wright argument used there exploits the quadratic structure and removes the extra logarithmic factor.

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