Math Finance — II

Week — 1

Utility theory.

"How to make a choice under risk"

Game Flip a coin
- H $\longrightarrow$ get £0.
- T $\longrightarrow$ £10.

How much you will be willing to pay.

$$\mathbb{E}(\text{Value}) = \frac{1}{2} \times £0 + \frac{1}{2}\, £10$$

$$= £5$$

New Game: (Bernoulli 1738)

Flip a coin
- $\frac{1}{2}$ H $\rightarrow$ £2
- $\frac{1}{2}$ T
  - $\frac{1}{2}$ H $\rightarrow$ £2^2
  - $\frac{1}{2}$ T
    - H $\longrightarrow$ £2^3
    - T
      - $\longrightarrow$ H
      - T ....

$$\mathbb{E}(\text{Game}) = £2 \cdot \frac{1}{2} + £2^2 \frac{1}{4} + £2^3 \cdot \frac{1}{8} + \cdots$$

$$= 1 + 1 + \cdots = \infty$$

$\cancel{\mathbb{E}(\text{Value})} \longrightarrow \mathbb{E}(\text{utility})$

$\cancel{\mathbb{E}(G)} \qquad\qquad \mathbb{E}\big(U(G)\big)$

utility   Game.

More formally

We will denote U for utility fn.

$$U : [0,\infty) \longrightarrow \mathbb{R}$$
$$w \longmapsto U(w)$$

w → Income / wealth.

We want to max Exp utility :

$$\max_{\Theta} \mathbb{E}\big(U(L)\big) \qquad L = \text{lottery}.$$

Θ = set of all feasiable strategies.

Axioms of Utility func:

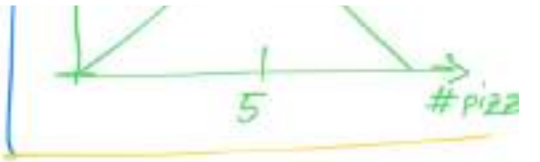
① More-is-better (Monotonicity).

w_1, w_2 — wealth level

$w_1 > w_2$

Side note:
Sometimes this breake down in practi

$$\Rightarrow U(\omega_1) \geq U(\omega_2).$$



② strictly increasing.

$$\omega_1 > \omega_2 \implies U(\omega_1) > U(\omega_2)$$

③ Completeness.

Bundle A, B, C.

$$A \succ B \;;\; B \succ A$$

$\succ$ denotes preference.

$$\Rightarrow \boxed{A \equiv B}$$

④ Mix-is-better (Convexity).

⑤ Rationality (Transitivity).

$$A \succ B, \; B \succ C$$

$$\Rightarrow \boxed{A \succ C}$$

⑥ Consistency

$A \succ B$. if there is another bundle C and $\alpha \in (0,1)$

$$\alpha A + (1-\alpha) C \succ \alpha B + (1-\alpha) C$$

Under the axion 1-6 :

Theorem:

If any preference satifies $\boxed{Ax\ 1-6}$ then

$\exists$ a real valued func^{U} (unique upto affine
(there exists)
transformation) ; and defined on the all possiable bundles such that :

Any two bundles A and B.

$$A \succ B \iff \mathbb{E}(U(A)) \geq \mathbb{E}(U(B)).$$

Afine transformation

U_1, U_2

$U_1 = a + b U_2$.

dup

Remark: Utility func are ordinal, and not cardinal.

If $U_2 = aU_1 + b$ with $a > 0$ then.

$$U_1 \equiv U_2.$$

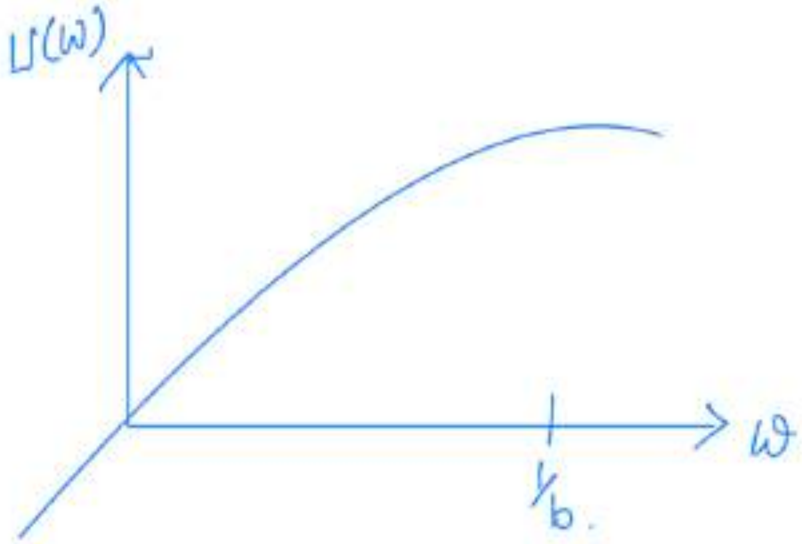
Utility function on Wealth.

$$U : [0,\infty) \longrightarrow \mathbb{R}$$
$$\omega \longmapsto U(\omega).$$

(can be thought of income)

Ex: Quadratic utility

$$U(\omega) = \omega - \frac{b}{2}\omega^2 \quad ; \quad \underline{b > 0} \text{ constant}.$$

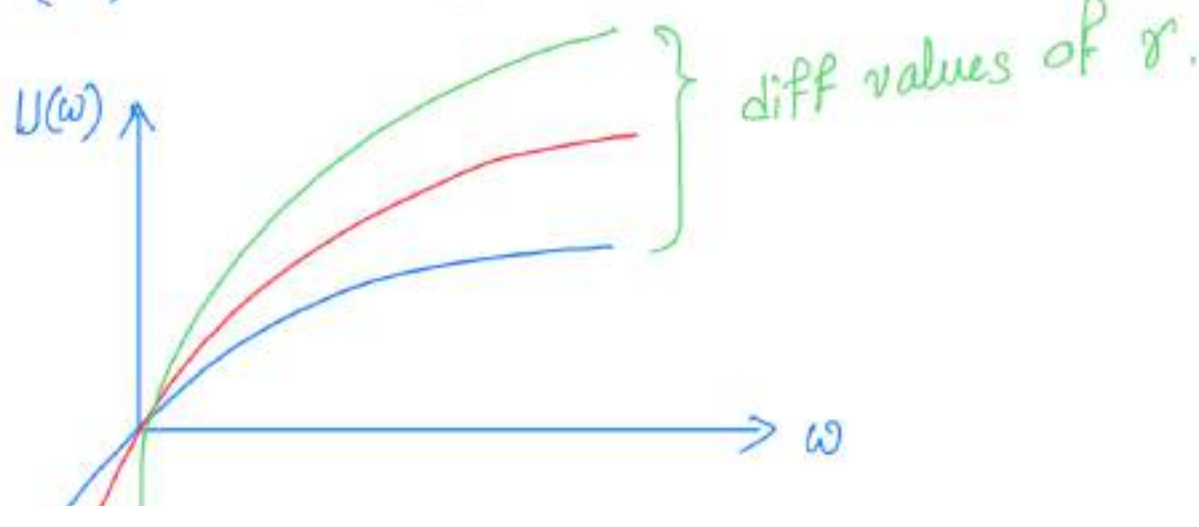


Not strictly increasing on $[0,\infty)$

But strictly increasing on $[0, 1/b]$

Ex: Exponential Utility

$$U(\omega) = -e^{-\gamma\omega} \quad \text{for } \gamma > 0$$



More difficult mathematically than Quadratic.

But
$$W \sim N(\mu, \sigma)$$
$\hookrightarrow$ End of period wealth.

$$\boxed{\mathbb{E}\big(U(W)\big) = -e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2}}$$

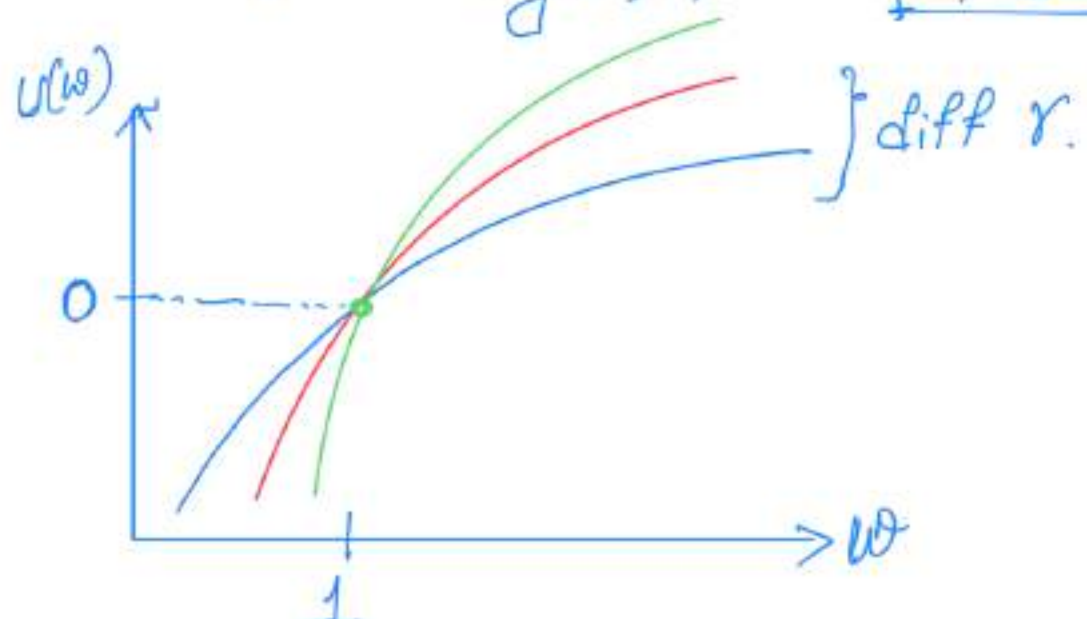
Sometime the exp utility is denoted as:
$$U(\omega) = \frac{1 - e^{-\gamma\omega}}{\gamma} \quad (HW)$$

Ex: Power Utility.

$$U(\omega) = \frac{\omega^{1-\gamma} - 1}{1-\gamma} \quad \text{for } \gamma > 0, \ \gamma \neq 1.$$

for $\gamma = 1$ by taking $\lim\limits_{\gamma \uparrow 1}$ we get

$$U(\omega) = \log(\omega); \quad \boxed{\omega > 0}$$



Risk-Aversion :

"A strictly risk-averse investor is unwilling to accept a fair game."

Notation:

W_0 = Wealth at time $t=0$

W_1 / W = Wealth at time $t=1$ / end of period wealth.

Def^n (fair game).

a RV   Not a RV

A game is fair $\iff \mathbb{E}(W) = \mathbb{E}(W_0) = W_0$

Example:

L lottery: start with W_0

$$W_1 = \begin{cases} W_0 + h_0 & \text{wp } p \\ W_0 + h_1 & \text{wp } (1-p) \end{cases}$$

When is the lottery fair:

We need to have

$$\mathbb{E}(W) = W_0$$

$$\Leftrightarrow (W_0 + h_0)p + (W_0 + h_1)(1-p) = W_0$$

$$\Leftrightarrow \boxed{h_0 p + h_1(1-p) = 0}$$

"A strictly risk-averse investor is unwilling to accept a fair game."

$$\Leftrightarrow \mathbb{E}(U(W)) < \mathbb{E}(U(W_0))$$

$$\Leftrightarrow \mathbb{E}(U(W)) < U(W_0)$$

def of fair game.

$$\Leftrightarrow \mathbb{E}(U(W)) < U(\mathbb{E}(W))$$

$$\Leftrightarrow p\,U(W_0 + h_0) + (1-p)\,U(W_0 + h_1) < U\left(p(W_0+h_0) + (1-p)(W_0+h_1)\right) \quad (*)$$

(*) has to be true for all fair gam.

$\Leftrightarrow$ (*) is true for any $h_0, h_1 \in \mathbb{R}$; and $p \in [0,1]$ with $p h_0 + (1-p) h_1 = 0$.

$\Leftrightarrow$ U has to be concave. (can also be seen using Jenson's ineq)

Lemma: A strict risk-averse investor always has concave utility func.

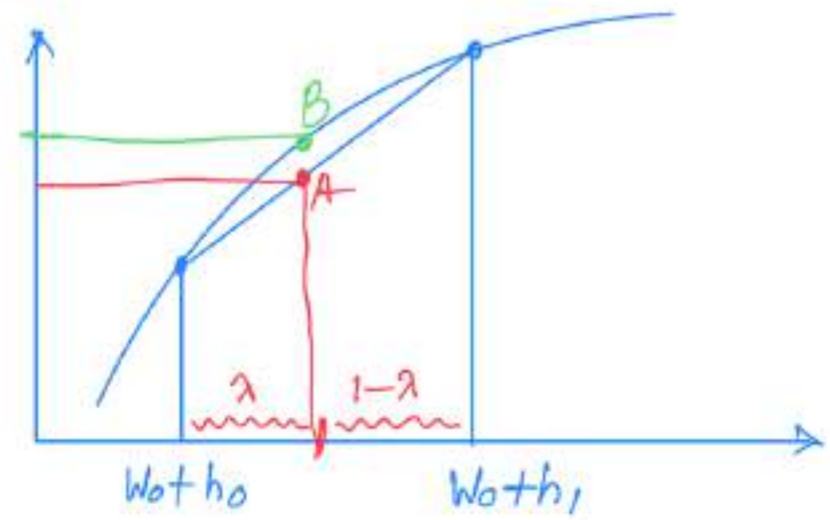
Def^n (Risk-aversion)

A investor is risk-averse iff the unitity fn is concave.

$$\Longrightarrow \boxed{\frac{d^2U}{dw} \leq 0 \quad ; \quad \frac{dU}{dw} > 0}$$

(concave). (strictly increasing U)

Risk-averse



B is above A

$$B: \; U\left(\lambda(W_0+h_0)+(1-\lambda)(W_0+h_1)\right)$$

$$A: \lambda U(W_0+h_0)+(1-\lambda)\,U(W_0+h_1).$$

---

two other type of investor.

Risk-seeking: $\dfrac{dU^2}{d^2w} \geqslant 0 \quad ; \quad \dfrac{dU}{dw} > 0$

(convex) (increasing).

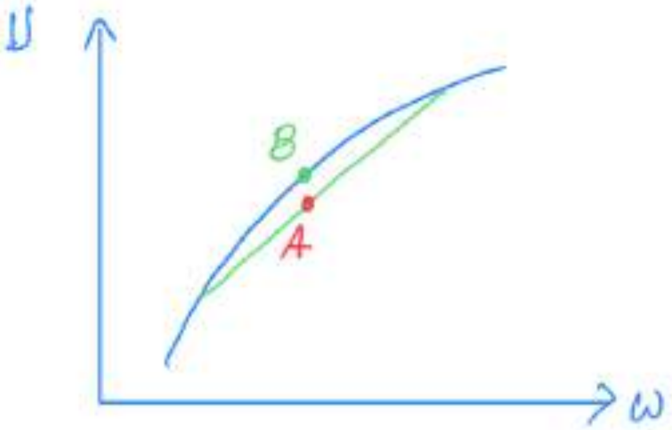
Risk – Neutral : $\frac{dU^2}{d\omega^2} = 0$; $\frac{dU}{d\omega} > 0$

(convex) (increasing).

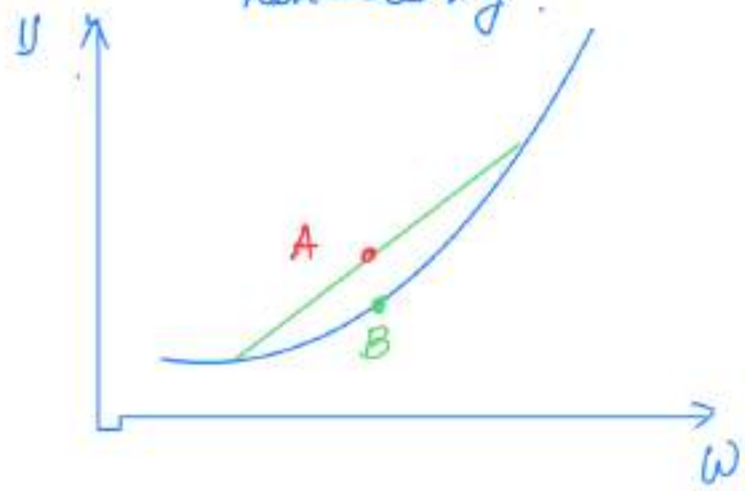
$\underbrace{\hspace{6cm}}$

linear utility fn.

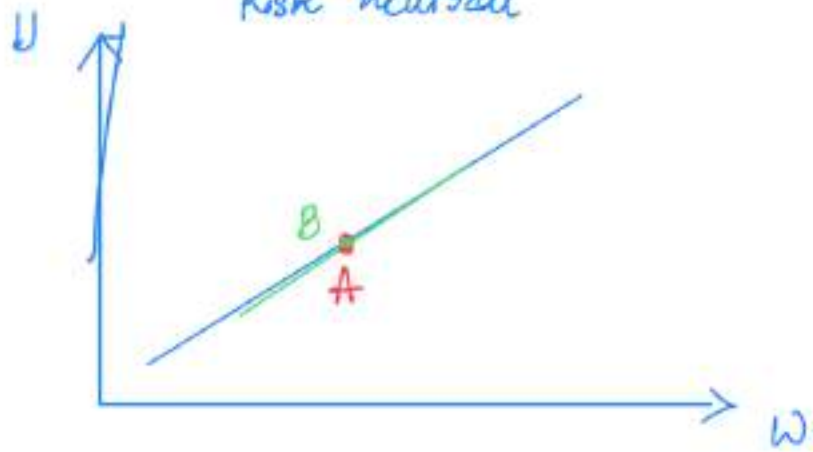
Risk averse



Risk – seeking.



Risk neutral



Math – Finance – II

Week – 2

Remainder:

Please submit the "Tutorial 1: suggestion" by tomorrow!

Note: Please use the Keats discussion forum / Q and A board for any qn/comments/feedback for the course (class/Hw/notes/tutorial)

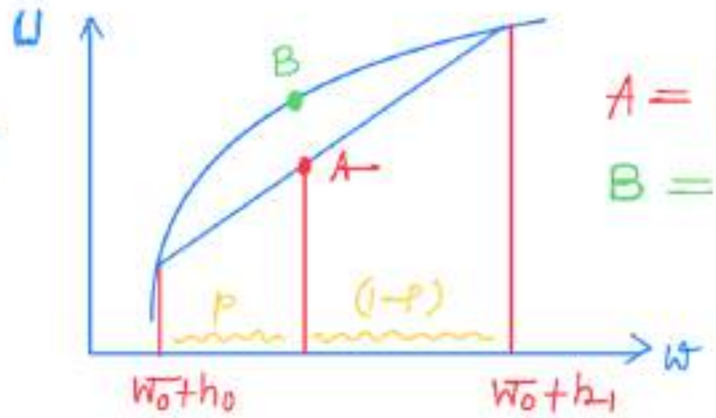
correction:

We want to maximize exp utility:

$$\max_{\theta} \mathbb{E}(L) \longrightarrow \text{should have been } \mathbb{E}(U(L))$$

Recap: A rational investor should max exp utility!

Risk-Averse:



$$A = pU(W_0+h_0)+(1-p)(W_0+h_1)$$

$$B = U(p(W_0+h_0)+(1-p)W_0+h_1)$$

Lottery:

start with W_0

$$\text{End of period wealth } W = \begin{cases} W_0+h_0 & \text{wp } p \\ W_0+h_1 & \text{wp } 1-p. \end{cases}$$

Def^n: (Marginal utility):

Marginal utility at a given wealth level ω^* is the slope of utility func at ω^*

$$\left.\frac{dU(\omega)}{d\omega}\right|_{\omega^*}$$

Notation: A diminishing marginal utility means utility func increases at a decreasing rate.

$$\left.\begin{array}{l} U' > 0 \\ U'' < 0 \end{array}\right\} \text{— Risk-averse.}$$

Notation: Increasing marginal utility.

$$\left.\begin{array}{l} U' > 0 \\ U'' > 0 \end{array}\right\} \text{— Risk-seeking.}$$

Motivation:

	Choice ①	choice ②
	⇓	⇓
W_0	$W^{①}$	$W^{②}$

The investor has utility func U.

$$\mathbb{E}\, U(W^{①}) \geq \mathbb{E}\, U(W^{②})$$

say: $\mathbb{E}\, U(W^{①}) > \mathbb{E}\, U(W^{②})$ $\rightarrow W_c^{①}$, $W_c^{②}$

$\Longrightarrow$ We prefer choice ① over choice ②.

"We can not say how much we prefer ① over ②."

Goal: We want a monetary unit; to say how much we prefer ① over ②.

Certainly Equivalent:

For any distribution of end-of-period wealth $\tilde{W}$, and for any valid utility func U, $\exists!$ constant (there exists unique) W_c st

$$U(W_c) = \mathbb{E}(U(\tilde{W}))$$

Note: W_c has same unit (say £) with W_0 and $\tilde{W}$.

Now we can we prefer choice ① over ② by

£ $W_c^{①}$ − £ $W_c^{②}$.

Lemma: For a risk averse investor $W_c \leq \mathbb{E}(\tilde{W})$.

Proof: Since this is a risk averse person:

$$\Rightarrow \mathbb{E}(U(\tilde{W})) \leq U(\mathbb{E}(\tilde{W})).$$

Hint: think of U to be concave and apply Jensen's Inequality.

$$\Rightarrow U(W_c) \leq U(\mathbb{E}(W))$$

$$\overset{?}{\Rightarrow} W_c \leq \mathbb{E}(W)$$ (this is true since U is strictly increasing). □

The opposit direction is in HW2.

Ex: $W_0 = £10$.

$$W = \begin{cases} £15 & \text{wp } 1/2 \\ £5 & \text{wp } 1/2 \end{cases}$$

The investor is risk averse.

Can you show $W_c \leq W_0$

Sol: $W_c \leq \mathbb{E}(W)$ (lemma)

$= 15 \times \frac{1}{2} + 5 \times \frac{1}{2}$

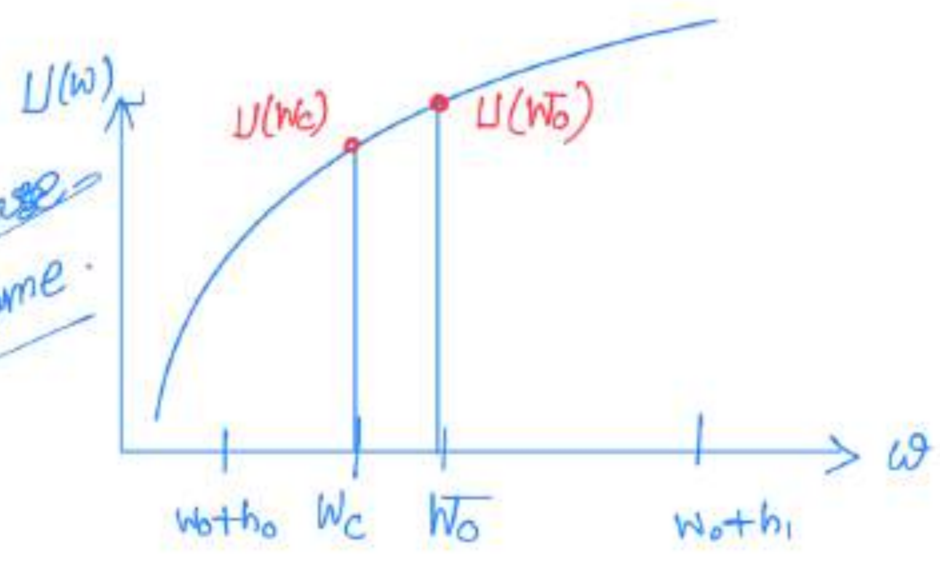
$= 10$

$= W_0$

Lemma: For a fair game ($\mathbb{E}(W) = W_0$); and risk-averse person one has:

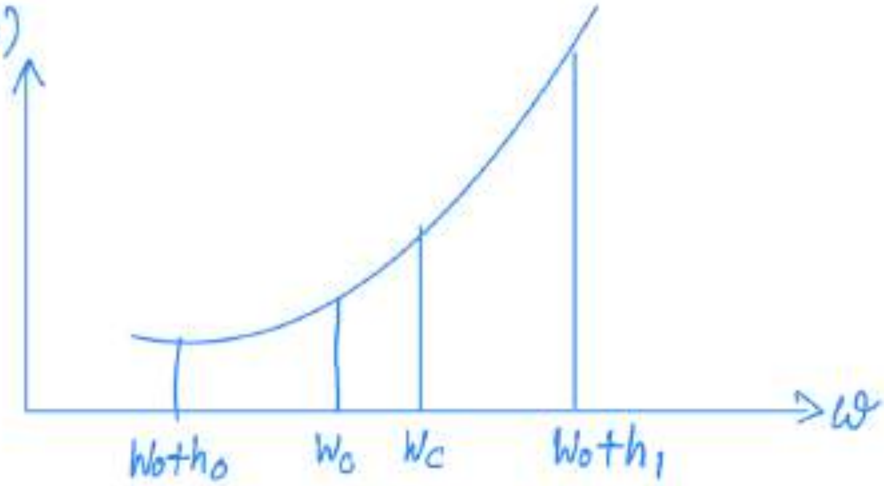
$$\boxed{W_c \leq W_0}$$

Risk-averse + Fair game.



Similarly:

Risk-seeking + Fair game.



Def^n: (Risk-premium)

$=$ the difference between exp. monetary value of the lottery and the certainty equivalent.

Mathematically: $\underbrace{R_p}_{\text{Risk-premium}} := \mathbb{E}(W) - W_C$

Remark: The risk-premium is the max amount a investor is willing to pay to avoid takig risk.

Remark:

Risk-averse : $R_p \geq 0$

Risk-seeking : $R_p \leq 0$

Risk-neutral : $R_p = 0$.

Coefficient of Risk-aversion.

Def^n The absolute risk-aversion (ARA) coefficient is defined as.

$$\boxed{\gamma^a(\omega) = -\frac{U''(\omega)}{U'(\omega)}}$$

Ex: $U(\omega) = -e^{-\bar{\gamma}\omega}$

$\Rightarrow \gamma^a(\omega) = \bar{\gamma}.$ ← exponencial utity is const.-abs. risk aversion func.

Def^n: The relative risk-aversion coefficient is given as:

$$\boxed{\begin{aligned}\gamma^r(\omega) &= -\omega\frac{U''(\omega)}{U'(\omega)} \\ &= \omega\gamma^a(\omega)\end{aligned}}$$

Ex: The power utility $\Rightarrow \gamma^r(\omega) \equiv$ const.

Example 6. *Consider an investor with the option of buying shares of an asset. The investor's initial wealth is W_0 and each share costs P_0. If the investor buys θ shares, the end-of-period wealth is $W = W_0 - \theta P_0 + \theta P_1$. Assume further $P_1 \sim \mathcal{N}(P_0 + \mu, \sigma)$ and that the investor uses exponential utility. What is the optimal number of shares to purchase?*

W_0 = initial wealth.

P_0 = stock price now.

P_1 = stock price at end of wealth period

$$\boxed{P_1 \sim \mathcal{N}(P_0+\mu, \sigma)}$$

$$W = W_0 - \theta P_0 + \theta P_1$$

θ = amount of share we buy.

Goal: find the optimal θ.

$$\boxed{U(w) = -e^{-\gamma w}}$$

$$\mathbb{E}(U(W)) = \mathbb{E}\left(U\left(W_0 - \theta P_0 + \theta P_1\right)\right)$$

$$= \mathbb{E}\left(U\left(W_0 - \theta P_0 + \theta\left(P_0 + \mu + \sigma Z\right)\right)\right)$$

where $Z \sim N(0,1)$.

$$= \mathbb{E}\left(-e^{-\gamma(W_0 - P_0\theta + \theta P_0 + \theta\mu + \theta\sigma Z)}\right)$$

$$= -e^{-\gamma W_0 - \gamma\theta\mu}\,\mathbb{E}\left(e^{-\gamma\theta Z}\right)$$

$$\underset{(MGF)}{=} -e^{-\gamma W_0 - \gamma\theta\mu} \times e^{\frac{\gamma^2\theta^2\sigma^2}{2}} \quad \text{— ①}$$

We want to find θ^* st. ① is maximized.

$$\theta^* = \underset{\theta}{\arg\max}\left[-e^{-\gamma W_0 - \gamma\theta\mu} \times e^{\frac{\gamma^2\theta^2\sigma^2}{2}}\right]$$

$$= \underset{\theta}{\arg\min}\left[e^{-\gamma W_0 - \gamma\theta\mu} \times e^{\frac{\gamma^2\theta^2\sigma^2}{2}}\right]$$

$$= \underset{\theta}{\arg\min}\left[e^{-\gamma W_0 - \gamma\theta\mu + \frac{\gamma^2\theta^2\sigma^2}{2}}\right]$$

$$= \underset{\theta}{\arg\min}\left[-\gamma W_0 - \gamma\theta\mu + \frac{\gamma^2\theta^2\sigma^2}{2}\right]$$

$$= \underset{\theta}{\operatorname{argmin}} \left[\underbrace{-\gamma\theta\mu + \frac{\gamma^2\theta^2\sigma^2}{2}}_{\text{quadratic on } \theta.} \right]$$

$$\boxed{\theta^* \overset{(HW)}{=} \frac{\mu}{\gamma\sigma^2}}$$

Recall:

$$\boxed{P_1 \sim \mathcal{N}(P_0 + \mu, \sigma)}$$

if $\mu > 0 \implies$ investor buyes the asset.

$\mu = 0 \implies \theta^* = 0$ (on an average $\mathbb{E}(P_1) = P_0$

$\implies$ fair game.

exp utility $\implies$ risk averse.

$\implies$ We don't accept fair game.

$\implies \theta^* = 0$)

Week – 3

Math - Finance — II

Remainder:

Please submit "tutorial 2 : suggestion" by tomorrow.

Jan 23

Utility fn on multiple goods:

Previously : $U : \mathbb{R}_+ \longrightarrow \mathbb{R}$

Now : $U : \mathbb{R}_+^n \longrightarrow \mathbb{R}$

Notation:

$\longrightarrow$ $P_1, \ldots P_n$ — Price for good i

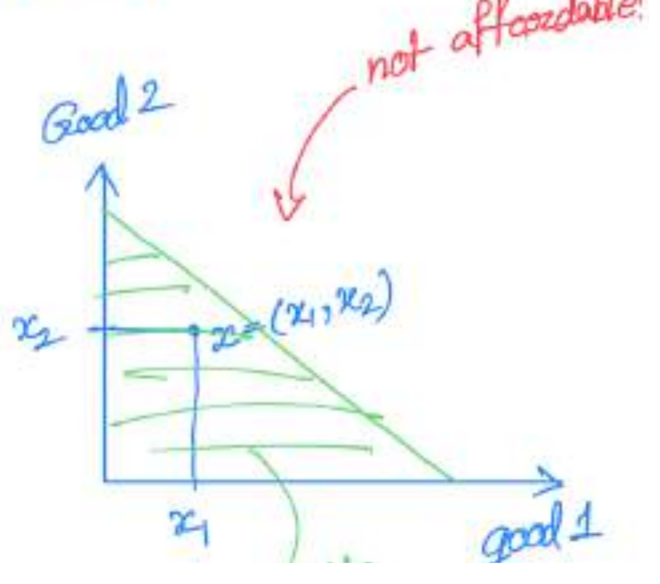
$\longrightarrow$ $x_1, \ldots x_n$ — quantity of good i

$\longrightarrow$ I / m — the total income / budget

Def^n

A consumer bundle is a vector:

$$x = (x_1, \ldots x_n) \in \mathbb{R}^n.$$



Affordable:

$x_1 P_1 + x_2 P_2 \leqslant m$ or not.

Affordable.

Note: $x_i \in \mathbb{R}_+$ not just integer.

Jan 30

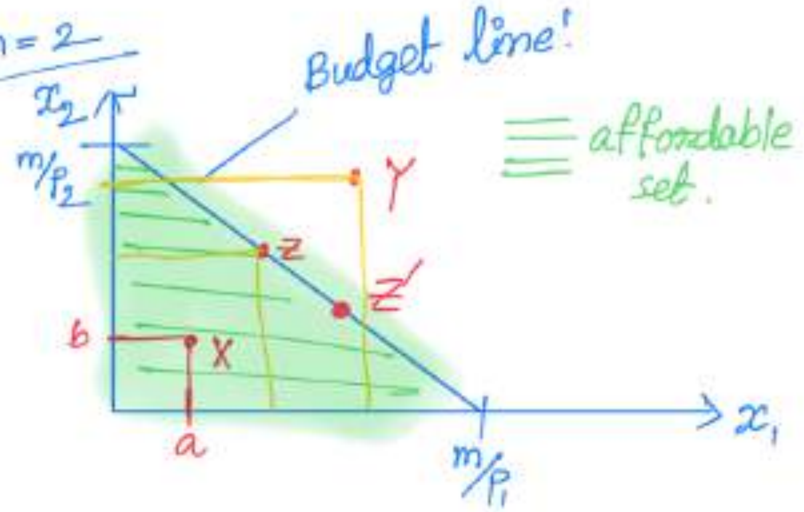
Budget Constrain : $p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m$.

Affordable consumption:

Bundle: $(x_1, \cdots x_n) \in \mathbb{R}^n_+$ st. $\underbrace{\sum_{i=1}^{n} p_i x_i}_{\text{total cost of bundle } (x_1, \cdots x_n)} \leq m$. ← income.

Budget line : $\boxed{p_1x_1 + \cdots + p_nx_n = m}$.

take $n=2$



Budget line:

$p_1x_1 + p_2x_2 = m$

$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$

slope $= -\frac{p_1}{p_2}$

"More is better" $\Longrightarrow Y \succ Z \succ X$

$\Rightarrow Y \succ X$

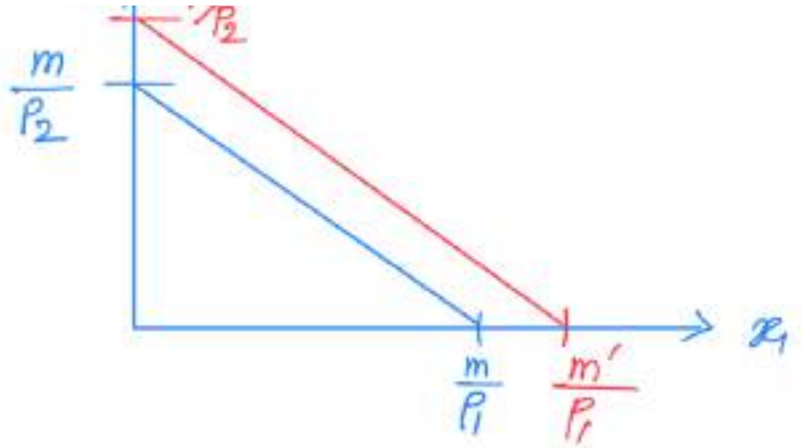
We can't afford Y
$Z \succ X$
$\Big\} \Longrightarrow$ We chose Z.

if $\cancel{E}U(Z) > \cancel{E}U(Z')$ we chose Z.

Case-1

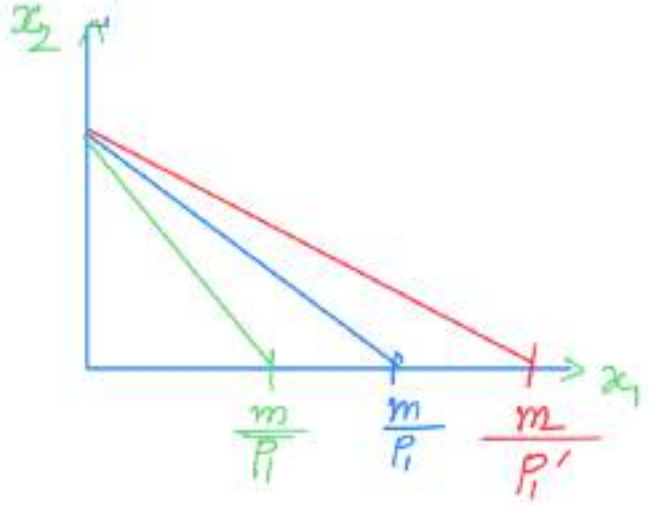
What happens when the budget/income ↑

x_2 m'



New income $m' > m$.

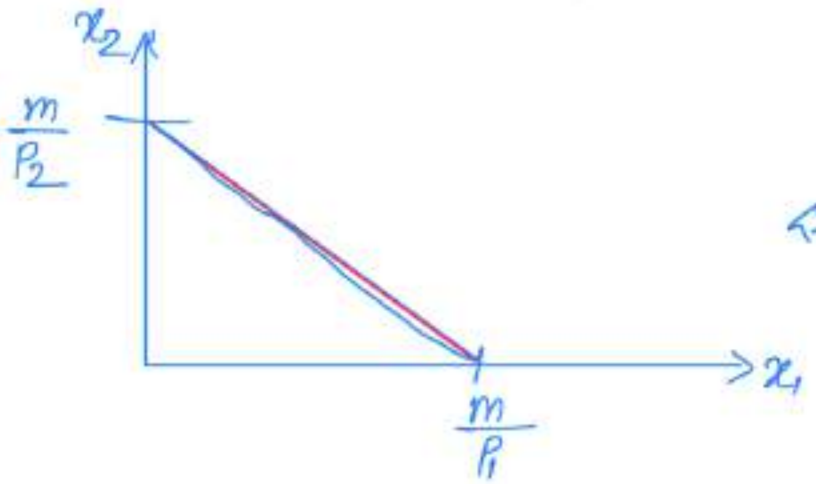
$p_1 \uparrow$ or $p_1 \downarrow$



$\overline{p_1} > p_1$

$p_1' < p_1$

What happens when there is a perfect inflation:



$$\sum p_i x_i = m$$

$$\iff \sum (\alpha p_i) x_i = \alpha m.$$

Budget line do not need to be straightline.

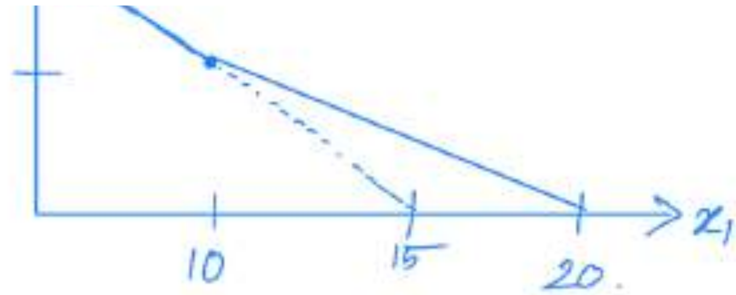
Ex: Quantity discount:

income $m = 30$

$p_2 = 2$

$$p_1 = \begin{cases} 2 & \text{for } x_1 < 10 \\ 1 & \text{for } x_1 \geq 10 \end{cases}$$

if $x_1 > 10$ we still pay £2 for the first 10 unit



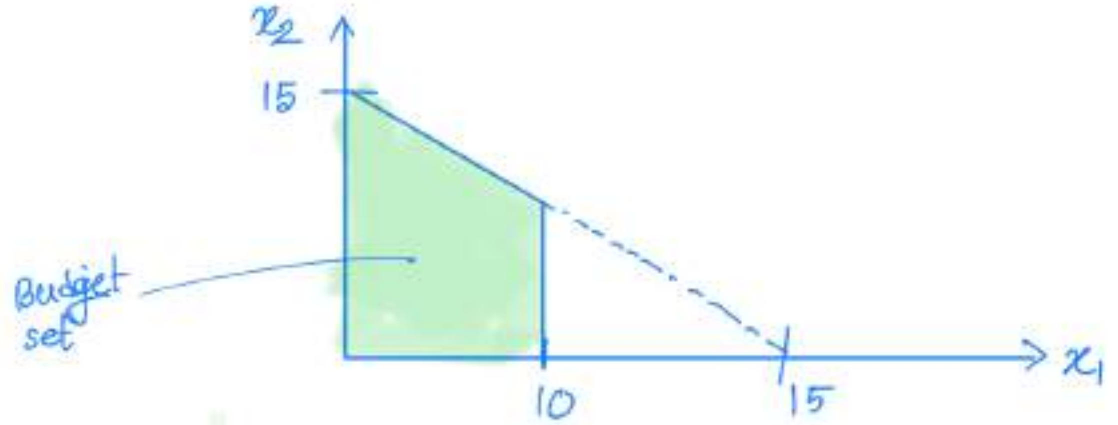
Ex:

Rationing

$m = 30$

$P_1 = 2 \longrightarrow$ One is not allowed to buy good 1 more than 10 unit.

$P_2 = 2$

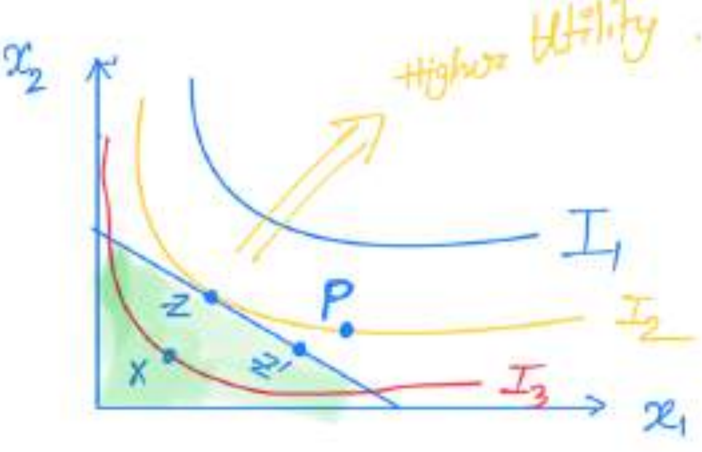


Defn (Indifference Curves)

This is the set of all bundles which consumer is indifferent to.

Let's denote The IC by I_1

$$\left.\begin{array}{l}(x_1 \cdots x_n) \in I_1 \\ (\overline{x_1}, \cdots \overline{x_n}) \in I_1\end{array}\right\} \Longrightarrow U((x_1 \cdots x_n)) = U(\overline{x_1}, \cdots \overline{x_n}).$$

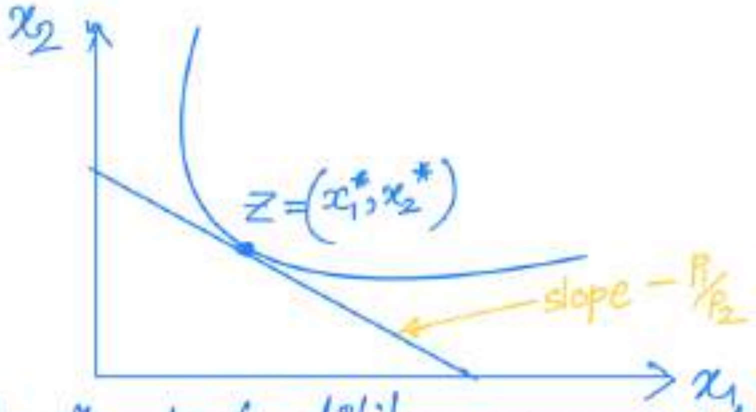


I_1 gives us bundles with more utility than I_2.

I_1 is not affordable.

Gole: To find the optimal bundle given $(P_1, \ldots P_n, m)$.

The optimal bundle should be tangent point of IC and budget set.



Z gives the optimal bundle.

Z represents the best utility you can get given m.

Note: • the optimal bundle can not be interior of Budget set.

• Tengency condition:

$$MRS\,(x_1^*, x_2^*) = \text{abs slope of budget line.}$$

(Marginal rate of substitution

The MRS is defined as

$$MRS\Big|_{(x_1,x_2)} = \frac{\frac{dU}{dx_1}}{\frac{dU}{dx_2}} = \frac{\text{Marginal utility of } x_1}{\text{Marginal utility of } x_2}.$$

$$\Rightarrow \boxed{\underbrace{\frac{\frac{dU}{dx_1}}{\frac{dU}{dx_2}}}_{MRS}\Bigg|_{\text{at optimal } := (x_1^*, x_2^*)} = \frac{P_1}{P_2}.}$$

for the n goods:

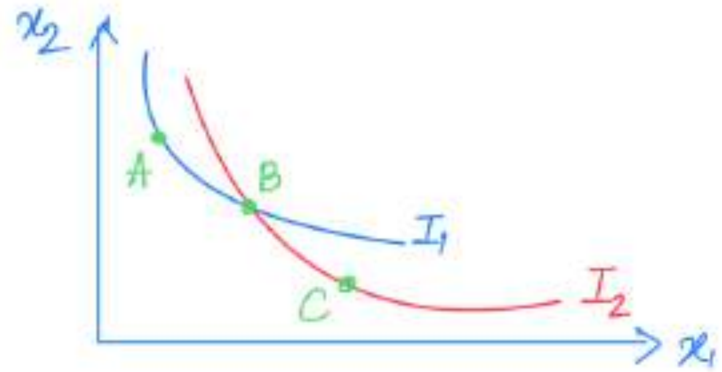
for the n goods:

$$\boxed{\frac{1}{P_1}\frac{dU}{dx_1} = \frac{1}{P_2}\frac{dU}{dx_2} = \cdots = \frac{1}{P_n}\frac{dU}{dx_n} = \lambda}$$

" Marginal utility per pound spend is same for all goods.

Lemma: The two IC can never intersect

Idea of proof:



$U(A) = U(B)$

$U(B) = U(C)$

transitivity:

$A \sim C$

Ex: $U(x_1, x_2) = \sqrt{x_1 x_2}$.

$U(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$

$U(x_1, x_2) = \ln x_1 + \frac{4}{5} \quad x_2^2$

Cobb-douglas utility fn:

$$U(x_1, x_2) = A x_1^{\alpha} x_2^{(1-\alpha)} \qquad \alpha \in (0,1)$$
$$A > 0$$

Mathematical formulation (Primal problem)

$$(P) \qquad \max_{x_1, \ldots x_n} U(x_1, \ldots x_n)$$

$\text{st} \quad p_1x_1 + p_2x_2 \cdots p_nx_n \leq m$

Lagrange Formulation:

$$\mathcal{L}(x_1, \dots x_n, \lambda) = U(x_1 \dots x_n) + \lambda(m - p_1x_1 - p_2x_2 \cdots - p_nx_n)$$

Solving $P \iff \max\limits_{x_1, \dots x_n, \lambda} \mathcal{L}$

Now we take derivative of $\mathcal{L}$ and equate to 0.

FOC:

① $\dfrac{\partial \mathcal{L}}{\partial x_i} = 0 \iff \dfrac{\partial U}{\partial x_i} - \lambda p_i = 0 \quad \forall i = 1 \dots n.$

② $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff p_1x_1 + \cdots p_nx_n = m.$

$\dfrac{1}{p_i}\left(\dfrac{\partial U}{\partial x_i}\right) = \lambda$

We have $n+1$ unknown, $n+1$ equation $\Longrightarrow (x_1^*, x_2^* \cdots x_n^*, \lambda^*)$.

What does this λ^* means:

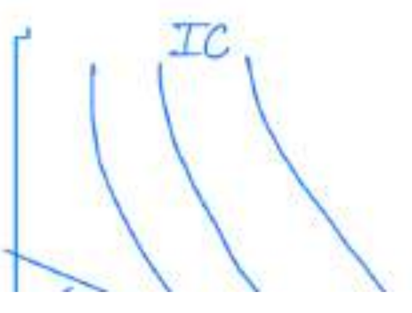
$\lambda^* = \dfrac{dU}{dx_i} \times \dfrac{1}{p_i}$

$\lambda^* = \dfrac{dU}{dm}$

λ^* is known as shadow price.

"Additional utility one can get by relaxing budget by 1 unit.

What about:



← corner solution.

"We will always assume the optimal solution $(x_1^* \dots x_n^*)$ is an interior solution"

Indirect utility fn:

This gives a function which tells us what is the max utility on can get given $(P_1 \cdots P_n, m)$

$$V(P_1, \cdots P_n, m) = U(x_1^*, \cdots x_n^*, m)$$

$$= U\left(x_1^*(P_1 \cdots P_n, m),\ x_2^*(P_1 \cdots P_n, m) \cdots\ x_n^*(P_1 \cdots P_n, m)\right)$$

Marshallian demand fn

this denotes the optimal quantity you consume for i^{th} good.

$$M_i(P_1 \cdots P_n, m) = x_i^* = x_i^*(P_1 \cdots P_n, m).$$

Properties of indirect utility:

1. Homogeneous of degree 0 in $(p_1, \cdots, p_n, m)$, i.e., $V(tp_1, \cdots, tp_n, tm) = V(p_1, \cdots, p_n, m)$.
2. Increasing in m, decreasing in each p_i.
3. Quasi-concave in $(p_1, \cdots, p_n, m)$.

Dual formulation of utility prob.

$\Longrightarrow$ Also known as Expenditure Minimizing problem.

$$(D) \quad \min_{(x_1, \cdots x_n)} \; P_1 x_1 + P_2 x_2 + \cdots P_n x_n .$$

Subject to $\quad U(x_1, \cdots x_n) \geq \overline{u}$

To solve this we again use Lagrangian Formulation:

$$\mathcal{L}(x_1 \cdots x_n, \lambda) = P_1 x_1 + \cdots + P_n x_n + \lambda\left(\overline{u} - U(x_1, \cdots x_n)\right)$$

$$\text{Solving } (D) \Longleftrightarrow \text{Solving } \min_{(x_1, \cdots x_n, \lambda)} \mathcal{L}(x_1, \cdots x_n, \lambda)$$

We do the same step as (P).

The first order condition (FOC)

$$① \; \frac{\partial \mathcal{L}}{\partial x_i} = 0 \Longleftrightarrow P_i - \lambda \frac{dU}{dx_i} = 0 \quad \forall i = 1, \cdots n.$$

$$② \; \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Longleftrightarrow \overline{u} = U(x_1, \cdots x_n).$$

Week — 4

Math Finance — II

Remainder: Please submit "Tutorial 3 suggession" by tomorrow.

$$\mathcal{L}(x_1 \cdots x_n, \lambda) = \underbrace{P_1 x_1 + P_2 x_2}_{} + \cdots + P_n x_n + \lambda\left(\bar{u} - U(x_1, \cdots x_n)\right)$$

FOC:

① $\dfrac{\partial \mathcal{L}}{\partial x_i} = 0 \implies P_i - \lambda \dfrac{\partial}{\partial x_i} U = 0$

$\forall i = 1, \cdots n$

$$\implies \frac{1}{P_i} \frac{\partial U}{\partial x_i} = \lambda$$

② $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies \bar{u} = U(x_1 \cdots x_n)$

① $\implies \dfrac{1}{P_i} \dfrac{\partial U}{\partial x_i} = \dfrac{1}{P_j} \dfrac{\partial U}{\partial x_j}$

Marginal utility for i

$$\iff \frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} = \frac{P_i}{P_j} \qquad \forall i, j \in \{1, \cdots n\}$$

Defn: Expenditure func:

min level of income/cost required to achive given level of utility $\bar{u}$.

$$e(p_1 \cdots p_n, \bar{u}) := \sum_{i=1}^{n} p_i \, x^h(p_1, \cdots p_n, \bar{u})$$

x^h is the solution of (D)

Def^n Hicksian Demand func

$$H_i(p_1 \cdots p_n, \bar{u}) = \text{the optimal level of good } i$$

we consume when price $(p_1 \cdots p_n)$ and we achive atleast utility leve $\bar{u}$.

$$:= x_i^h(p_1 \cdots p_n, \bar{u})$$

Ex: Cobb-Douglas utility.

$$U(x_1, x_2) = \sqrt{x_1 x_2}$$

Dual problem:

$$\min \; p_1 x_1 + p_2 x_2$$

$$\text{st} \quad U(x_1, x_2) \geq \bar{u} \iff \sqrt{x_1 x_2} \geq \bar{u}$$

$$\iff x_1 x_2 \geq \bar{u}^2$$

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda\big(\underbrace{\bar{u}^2 - x_1 x_2}_{\bar{u} - \sqrt{x_1 x_2}}\big)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \iff P_1 - \lambda x_2 = 0 \iff x_2 = \frac{P_1}{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \iff P_2 - \lambda x_1 = 0 \iff x_1 = \frac{P_2}{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff \bar{u}^2 = x_1 x_2$$

$$① + ② \implies \boxed{\frac{P_1}{x_1} = \frac{P_2}{x_2}} \; (= \lambda) \quad (*)$$

$x_2 = x_1 P_2/P_1$

Replacing (*) in ③ we get :

$$\rightarrow x_1\left(\frac{P_2 x_1}{P_1}\right) = \bar{u}^2$$

$$\implies \boxed{x_1^{**} = \sqrt{\bar{u}^2 \frac{P_2}{P_1}}}$$

This gives us the optimal level of good 1 we want to consume .

$$\boxed{\begin{aligned} x_2^{**} &= x_1^{**} \times \frac{P_2}{P_1} \\ &= \bar{u}\sqrt{\frac{P_1}{P_2}} \end{aligned}}$$

Hicksian Demand

$$H_1(P_1, P_2, \bar{u}) = \bar{u}\sqrt{\frac{P_2}{P_1}}$$

$$H_2(P_1, P_2, \bar{u}) = \bar{u}\sqrt{\frac{P_1}{P_2}}$$

$$e(P_1, P_2, \bar{u}) = \sum P_i x_i^h$$

$$= P_1\left(\bar{u}\sqrt{\frac{P_2}{P_1}}\right) + P_2\left(\bar{u}\sqrt{\frac{P_1}{P_2}}\right)$$

$$= 2\bar{u}\sqrt{P_1 P_2}.$$

Primal—Dual relationship (interior sol).

- Links indirect utility with expenditure func.

Imp

$$\boxed{\begin{aligned} \bar{u} &= V(P_1, P_2 \cdots P_n, m) \\ m &= e(P_1, \cdots P_n, \bar{u}) \end{aligned}}$$

- Links Marshallian and Hicksian demands

Imp

$$\boxed{\begin{aligned} M_i(P_1 \cdots P_n, m) &= H_i\left(P_1 \cdots P_n, V(P_1 \cdots P_n, m)\right) \\ H_i(P_1 \cdots P_n, \bar{u}) &= M_i\left(P_1 \cdots P_n, e(P_1 \cdots P_n, \bar{u})\right) \end{aligned}}$$

Sec 3.5 in notes

2025 May exam Qn.

1. (a) Bill is risk-neutral. What is the absolute risk aversion for Bill? How does he rank the following lotteries?

$$L1 = \begin{pmatrix} £24 & £12 & £48 & £6 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}, \qquad L2 = \begin{pmatrix} £180 & £0 & £90 \\ \frac{1}{20} & \frac{17}{20} & \frac{2}{20} \end{pmatrix}$$

(Note: The first line denotes the possible outcome of the lottery and the second line denotes the corresponding probability)

What are the risk premiums associated with lottery L1 and L2 respectively for Bill? **[10+10+10%]**

(b) Suppose an agent consumes three goods, $\{x_1, x_2, x_3\}$ and the corresponding prices are $\{p_1, p_2, p_3\}$ respectively. Her utility is $U(x_1, x_2, x_2) = \sqrt{x}_1 + \sqrt{x}_2 + \sqrt{x}_3$. The agent has income M.

(b-1) Calculate the indirect utility function. **[40%]**

(b-2) Write in detail the primal and the dual consumer problem associated with the above utility. **[30%]**

1. (a) Consider a lottery which has probability density $f(w) = \frac{2}{3}w$ for $1 \leq w \leq 2$. If the client has log utility (i.e. $U(w) = \log(w)$), calculate the expected utility (i.e. $\mathbb{E}(U(\text{Lottery}))$) and the certainty equivalence of this lottery. **[40%]**

(b) An agent chooses a consumption bundle $x = (x_1, x_2)$ to maximise her utility. Her utility function is

$$U(x_1, x_2) = \left(x_1 - \frac{1}{2}x_1^2\right) + \left(x_2 - \frac{1}{2}x_2^2\right)$$

where we assume $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$. The agent has income $M = 1$. Suppose the prices are $p_1 = 3$ and $p_2 = 2$.

(b-1) Solve for the agent's optimal consumption. **[40%]**

(b-2) Furthermore, solve the agent's optimal consumption if her income doubles. **[20%]**

2025 Aug exam Qn.

Week – 4

Mathematical Finance II

Portfolio Optimization.

Markowitz in 1952.

"One should consider risk and return together.

Goal: Given an inetial capital $V(0)$ and N many securities : How we allocate the capital $V(0)$ so that the return of the portfolio is "optimal".

Assumptions:

① only concider single period model.

$\Rightarrow$ two time period

$t=0$ (inetial time) $\quad$ $t=1$ (terminal time).

② Each investment is measured by two quantity only

$\longrightarrow$ Mean

$\longrightarrow$ Var/sd. $\quad$ (this means we ignore more informative shape of the distribution)

Remark: for Normal dis; this gives full knowledge of the dist.

③ The risk of a portfolio is measured by sd.

$\boxed{\text{risk} \neq \text{var}}$ as we want the unit of risk to be same as unit of security.

Additional assumption:

- investor is rational
- No arbitrage.
- everyone has same information.
- market is liquid.
- no transaction cost
- no tax.
- same borrow/lend limit for all.

Defn: (short selling)

selling something which you do not own at time $t=0$

"Short selling is Risky"

short selling 1 unit of Stock A. $S(0)$ (= known)

We agree to sell at time $t=1$.

$S(1)$ is unknown:

The total loss/gain $= S(1) - S(0)$

	Stock A		Stock B	
	Position	Price	Position	Price.
$t=0$				

$t=0$	x_1	$S_1(0)$	x_2	$S_2(0)$
$t=1$	x_1	$S_1(1)$	x_2	$S_2(1)$

— known.

— RV.

Portfolio: P/V $= \begin{cases} x_1 \text{ unit of stock A} \\ x_2 \text{ unit of stock B.} \end{cases}$

Weights:

$$\omega_1 := \frac{x_1 S_1(0)}{V(0)}$$

$$\omega_2 := \frac{x_2 S_2(0)}{V(0)}$$

where $V(0) := x_1 S_1(0) + x_2 S_2(0)$

↳ the amount invested at $t=0$.

Prop:

- $\omega_1 + \omega_2 = 1$ $\quad \omega_1 + \cdots + \omega_n = 1$
- $\omega_i :=$ weights $\in \mathbb{R}$
  - $+ve$ — buying.
  - 0 — no position
  - $-ve$. — short selling.

(def^n) • The rate of return is given by $\boxed{K_i := \frac{S_i(1) - S_i(0)}{S_i(0)}}$

Lemma:
For a portfolio with weights $\omega_1 \cdots \omega_n$ and return $K_1 \cdots K_n \implies$ the portfolio return is given by

$$K_V = \omega_1 K_1 + \omega_2 K_2 + \cdots \omega_n K_n$$

Proof: Please see notes.

Risk and return of a portfolio

Defⁿ: (Exp return) this is nothing but

$$\mu_i \doteq \mathbb{E}(K_i) = \mathbb{E}\left(\frac{S_i(1) - S_i(0)}{S_i(0)}\right) = \frac{\mathbb{E}(S_i(1) - S_i(0))}{S_i(0)}$$

↳ exp return of security i.

Defⁿ: (Risk) the risk of security i is given by

$$\sigma_i = sd(K_i) = \sqrt{Var(K_i)}$$

$$= \sqrt{\mathbb{E}(K_i^2) - (\mathbb{E}(K_i))^2}$$

How to connect this to portfolio risk and exp return?

Lemma: The portfolio risk + exp return:

$$\mu_V = \omega_1 \mu_1 + \cdots \omega_n \mu_n$$

$\boxed{\text{Assume } n=2}$ $$\sigma_V = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \underbrace{Cov(K_1, K_2)}_{\rho_{K_1 K_2} \sigma_1 \sigma_2}$$

↳ the correlation coefficient between K_1 and K_2

(Sometimes written as ρ_{12})

Proof:

Proof

$$\sigma_V = sd(K_V)$$

$$\Leftrightarrow \sigma_V^2 = Var(K_V)$$

$$= Var(\omega_1 K_1 + \omega_2 K_2)$$

$$= \mathbb{E}\left((\omega_1 K_1 + \omega_2 K_2)^2\right) - \left\{\mathbb{E}(\omega_1 K_1 + \omega_2 K_2)\right\}^2$$

$$= \mathbb{E}\left(\omega_1^2 K_1^2 + \omega_2^2 K_2^2 + 2\omega_1\omega_2 K_1 K_2\right) - \left\{\omega_1 \underbrace{\mathbb{E}(K_1)}_{\mu_1} + \omega_2 \underbrace{\mathbb{E}(K_2)}_{\mu_2}\right\}^2$$

$$= \omega_1^2\left\{\mathbb{E}(K_1^2) - (\mu_1^2)\right\} + \omega_2^2\left\{\mathbb{E}(K_2^2) - \mu_2^2\right\} + 2\omega_1\omega_2\left(\mathbb{E}(K_1K_2) - \mu_1\mu_2\right)$$

$$= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2 Cov(K_1, K_2)$$

<u>Lemma:</u> If shortselling not allowed then: $\Rightarrow \omega_1, \omega_2 \geq 0$

$$\sigma_V \leq \max\{\sigma_1, \sigma_2\}.$$

<u>Proof:</u>

$$\sigma_V^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\underbrace{\omega_1\omega_2}_{+ve}\underbrace{\rho_{12}}_{|\rho_{12}|\leq 1}\underbrace{\sigma_1\sigma_2}_{+ve}$$

$$\leq \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_1\sigma_2$$

$$= (\omega_1\sigma_1 + \omega_2\sigma_2)^2$$

$$\leqslant \left(\omega_1 \max\{\sigma_1, \sigma_2\} + \omega_2 \max\{\sigma_1, \sigma_2\}\right)^2$$

$$= \left(\max\{\sigma_1, \sigma_2\}\right)^2 \left(\underbrace{\omega_1 + \omega_2}_{=1}\right)^2$$

$$= \left(\max\{\sigma_1, \sigma_2\}\right)^2$$

$$\Longleftrightarrow \quad \sigma_V \leqslant \max\{\sigma_1, \sigma_2\}.$$

Week - 5

Math - Finance - II

Remainder: Please submit "Tutorial-4 suggession" by tomorrow.

Note: Next week the class is on Wed (Not Friday)

Revisit of last class:

Lemma: If $\boxed{\text{shortselling not}}$ allowed then: ($\Rightarrow \omega_1, \omega_2 \geq 0$)

$$\boxed{\sigma_V \leq \max\{\sigma_1, \sigma_2\}}.$$

- σ_1, σ_2 say risk of stock A, B.
V - portfolio containing stock A, B.

Proof:

$$\sigma_V^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\underbrace{\omega_1\omega_2}_{+ve}\underbrace{\rho_{12}}_{|\rho_{12}|\leq 1}\underbrace{\sigma_1\sigma_2}_{+ve}$$

$$\leq \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_1\sigma_2 \cdot 1$$

$$= (\omega_1\sigma_1 + \omega_2\sigma_2)^2$$

wlog $\sigma_1 \leq \sigma_2$

$$\leq \left(\omega_1 \overbrace{\max\{\sigma_1,\sigma_2\}}^{\sigma_2} + \omega_2 \overbrace{\max\{\sigma_1,\sigma_2\}}^{\sigma_2}\right)^2$$

$$= \left(\overbrace{\max\{\sigma_1,\sigma_2\}}^{\sigma_2}\right)^2 (\omega_1+\omega_2)^2$$

$$= \Big(\underbrace{\max\{\sigma_1, \sigma_2\}}_{\sigma_2}\Big)^2$$

$$\Longleftrightarrow \boxed{\sigma_V \leqslant \max\{\sigma_1, \sigma_2\}}.$$

13th Feb The above phenomena is call diversification.

Take away: We can reduce risk by having 2 security over having 1 security.

Example 4. *Suppose that there are n assets, all of which are mutually uncorrelated, i.e., the return of each asset K_i is uncorrelated with that of any other asset K_j in the group. Suppose that the expected rate of return and variance of each of these assets is μ and σ^2, respectively. A portfolio is created by taking equal portions of each of these assets; that is, $w_i = 1/n$ for each i. Calculate the expected rate of return and the risk of this portfolio.*

We have n stocks

Each i^{th} stock :

$$\mu_i = \mu$$

$$\sigma_i = \sigma$$

$\Longrightarrow$ the stocks are uncorrelated

$$Cov(K_i, K_j) = 0 \quad \forall i \neq j$$

$$Cov(K_i, K_i) = Var(K_i) = \sigma_i$$

$$\text{Portfolio } V = \begin{pmatrix} \frac{1}{n} \text{ unit of stock } 1 \\ \vdots \\ \frac{1}{n} \text{ unit of stock } n \end{pmatrix}$$

$$\Rightarrow \omega_1 = \omega_2 = \cdots = \frac{1}{n}$$

what is the risk of the portpolio V.

$$\sigma_V = sd(K_V)$$

$$\Leftrightarrow \sigma_V^2 = Var(K_V)$$

$$= Var\left(\sum_{i=1}^{n} \omega_i K_i\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j Cov(K_i, K_j)$$

$$= \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{n} \sigma^2 \qquad \begin{cases} Cov(K_i, K_j) = 0 \quad \forall i \neq j \\ Cov(K_i, K_i) = Var(K_i) = \sigma_i \end{cases}$$

$$= \frac{1}{n} \sigma^2$$

$$\boxed{\sigma_V = \frac{1}{\sqrt{n}} \sigma} \xrightarrow[\text{theoritically.}]{\text{as } n \to \infty} 0.$$

individual risk.

Can we always make the (risk of V) $\longrightarrow 0$ by chosing appropriate weights.

$\Rightarrow$ No.

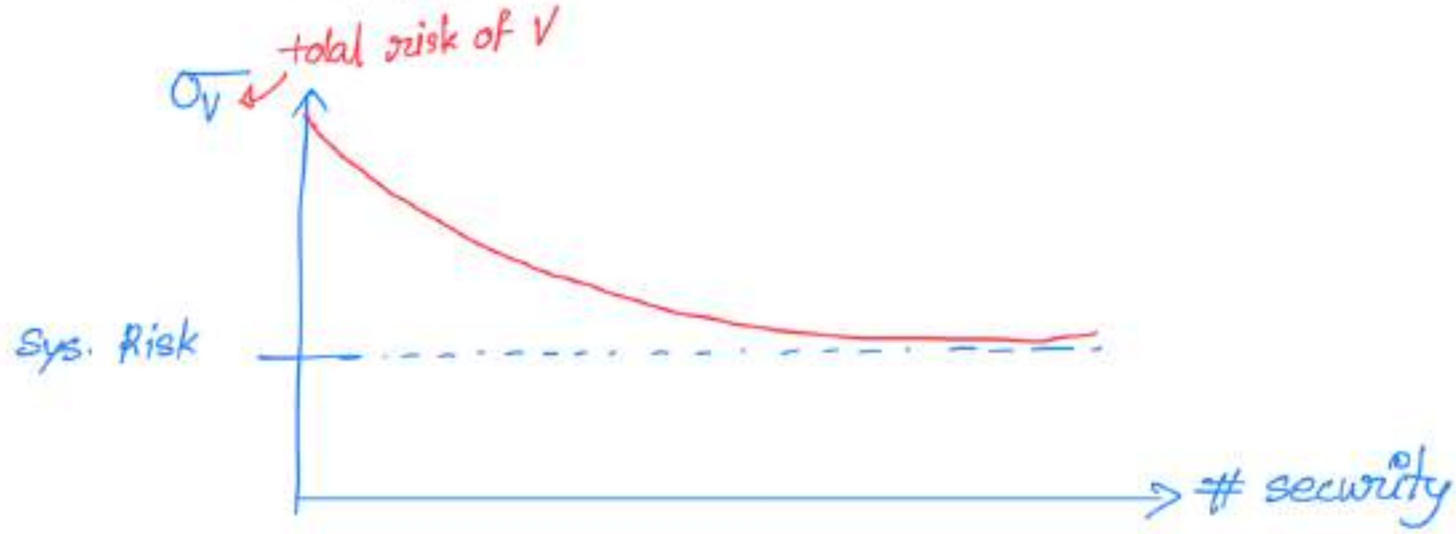
Type of Risk.

$\rightarrow$ Diversifiable risk : (Ex: Internal/external buisness risk, financial risk (company specific).

$\hookrightarrow$ We can make this risk $\downarrow 0$ by chossing correct portfolio weights.

$\rightarrow$ Systematic risk. (Ex: Market risk, interest rate risk, Inflation risk)

$\hookrightarrow$ can not be diversified.



Ex 5 of lecture note:

- n stock

weights $\frac{1}{n}$ for each stock i

each stock i has

$$\mu_i = \mu$$

$$\sigma_i = \sigma$$

$$\sigma_V^2 = Var(K_V)$$

$$= Var\left(\sum_{i=1}^{n} w_i K_i\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \, Cov(K_i, K_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2} \, Cov(K_i, K_j)$$

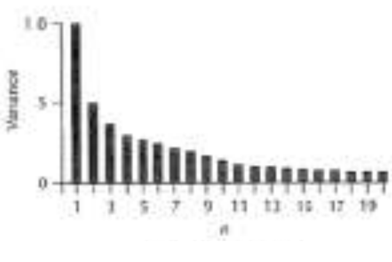
$$= \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 + \sum_{i\neq j=1}^{n} \frac{1}{n^2} \, \underbrace{Cov(K_i, K_j)}_{\rho_{ij}\sigma_i\sigma_j \,=:\, \sigma_{ij}}$$

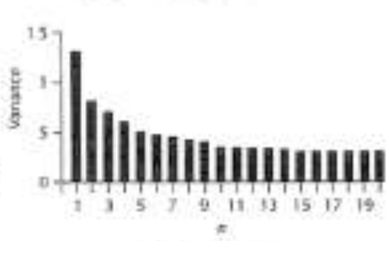
$$= \frac{1}{n}\underbrace{\left(\sum_{i=1}^{n} \frac{1}{n}\sigma^2\right)}_{\text{avg variance}} + \frac{n^2-n}{n^2}\underbrace{\left(\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n} \frac{1}{n^2-n}\sigma_{ij}\right)}_{\text{Avg covariance.}}$$

$$\text{avg variance} \xrightarrow{n\to\infty} 0 \qquad\qquad \text{Avg covariance} \longrightarrow \text{Avg cov.}$$

Avg cov. ↳ systomatic risk.

Homework: Show that when all the assets are positively correlated and at least two of the assets are not perfectly correlated, the risk of the portfolio is still lower than σ but higher than the uncorrelated case. Also, show that if the assets are negatively correlated, then the risk decreases even further from the uncorrelated case.





(a) Uncorrelated assets (b) Correlated assets

FIGURE 6.7 Effects of diversification. If assets are uncorrelated, the variance of a portfolio can be made very small. If assets are positively correlated, there is likely to be a lower limit to the variance that can be achieved.

Let's go back to $n=2$.

Goal: Find the 'best' portfolio weights.

The portfolio risk σ_V has a formula.

$$\sigma_V^2 = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1\omega_2 \underbrace{\sigma_1 \sigma_2 \rho_{12}}_{\text{pre fixed}}$$

$$\omega_1 + \omega_2 = 1 \Rightarrow \omega_2 = 1-\omega_1$$

$$\sigma_V^2 = \omega_1^2 \sigma_1^2 + (1-\omega_1)^2 \sigma_1^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{12}.$$

$$f: \omega_1 \longmapsto \underbrace{\omega_1^2 \sigma_1^2 + (1-\omega_1)^2 \sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{12}}_{\sigma_V^2}.$$

Case-1 $\rho_{12} = +1$

say we want $\sigma_V \equiv 0 \Leftrightarrow \sigma_V^2 = 0$ (1)

$$\Leftrightarrow \omega_1^2 \sigma_1^2 + (1-\omega_1)^2 \sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\cancel{\rho_{12}} = 0$$

$$\Leftrightarrow \left(\omega_1\sigma_1 + (1-\omega_1)\sigma_2\right)^2 = 0$$

$$\Leftrightarrow \left|\omega_1\sigma_1 + (1-\omega_1)\sigma_2\right| = 0$$

$$\boxed{\sigma_V = 0} \Longleftrightarrow \boxed{\text{if } \sigma_1 \neq \sigma_2 \text{ then } \omega_1 = \frac{-\sigma_2}{\sigma_1-\sigma_2},\ \omega_2 = \frac{\sigma_1}{\sigma_1-\sigma_2}}$$

→ We always need shortselling.

Case 2 $\rho_{12} = -1$

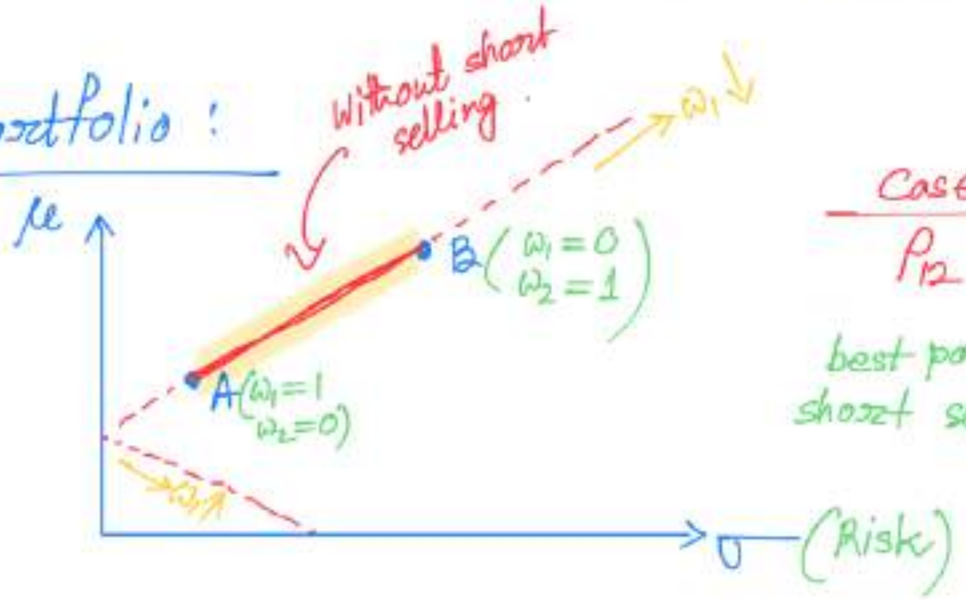
then $\sigma_V = |\omega_1\sigma_1 - \omega_2\sigma_2|$

Furthermore,

$$\boxed{\sigma_V = 0} \iff \boxed{\omega_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}, \quad \omega_2 = \frac{\sigma_1}{\sigma_1+\sigma_2}}$$

→ We do not need shortselling for the best portfolio

Draw portfolio:

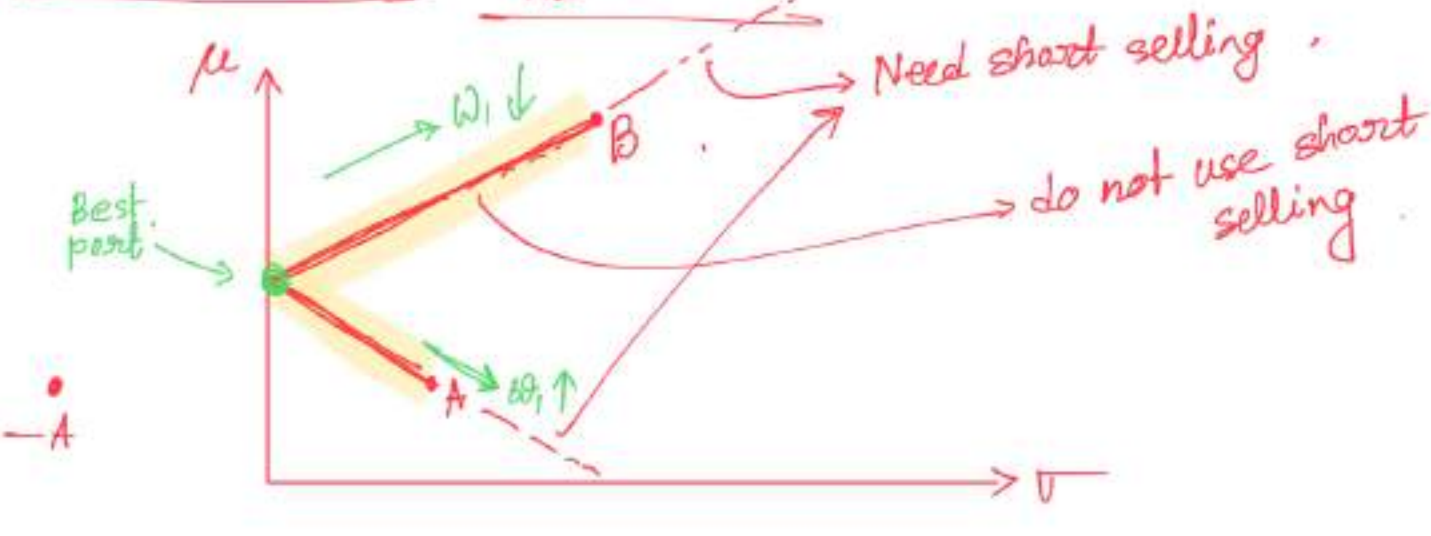


Case-1
$\rho_{12} = +1$.

best port without short selling $= \left\{\begin{matrix}\omega_1=1\\ \omega_2=0\end{matrix}\right\}$

We start from security A and B.
then each portfolio constructed from A and B can be uniquely represented in $\sigma - \mu$ plane.

For case 2 $\rho_{12} = -1$



Case − 3 $\quad \rho_{12} \in (-1, +1)$

$$f : \omega_1 \longmapsto \sqrt{\sigma_1 \omega_1^2 + \sigma_2 (1-\omega_1)^2 + 2\sigma_1 \sigma_2 \omega_1 (1-\omega_1) \rho_{12}}$$

$\Rightarrow$ Equiv func :

$$g : \omega_1 \longmapsto \sigma_1 \omega_1^2 + \sigma_2 (1-\omega_1)^2 + 2\sigma_1 \sigma_2 \omega_1 (1-\omega_1) \rho_{12}$$

Need to take g'

$$\Rightarrow g' \equiv 0$$

$$\Rightarrow \omega_1^* \; ; \; \omega_2^* = 1 - \omega_1^*$$

We can also say $g'' > 0$.

Theorem: if $\rho_{12} \in (-1, +1)$ then the σ_V^{min} may not be $\equiv 0$.

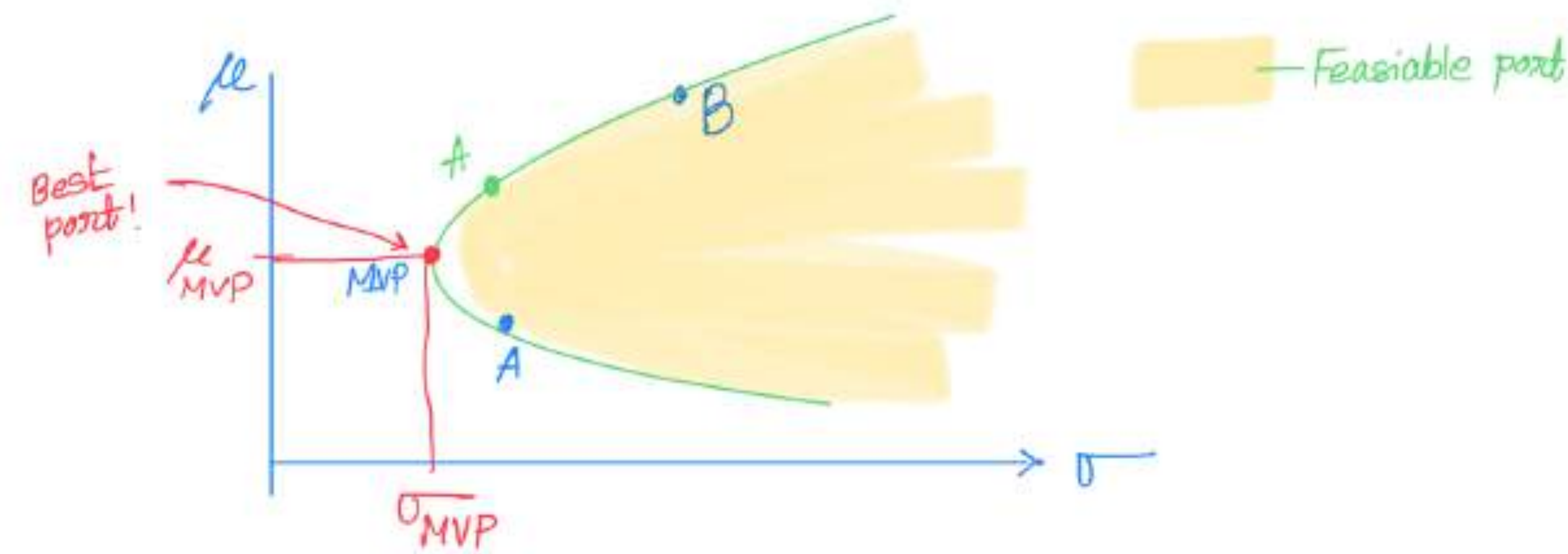
Furthermore the weight of σ_V^{min} is given as.

$$\left.\begin{aligned} w_1^{min} &= \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}} \\ w_2^{min} &= \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}} \end{aligned}\right\}$$

short selling is allowed.

If we do not allow short selling:

$$\left(\omega_1^{min}, \omega_2^{min}\right) = \begin{cases} (0,1) & \omega_1^{min} < 0 \\ \left(\omega_1^{min}, \omega_2^{min}\right) & \text{if } 0 \leq \omega_1^{min} \leq 1 \\ (1,0) & \text{if } \omega_1^{min} > 1 . \end{cases}$$



Min Variance Port (MVP)

the portfolio with the smallest possiable risk among all feasiable portfolio.

Notation $\omega_{MVP} = \left(\omega_1^{MVP}, \omega_2^{MVP} \cdot ----, \omega_N^{MVP}\right)$

Def^n (feasiable set)

The set of all portfolio which can be constructed by investing ONLY in security 1, ---, security N.

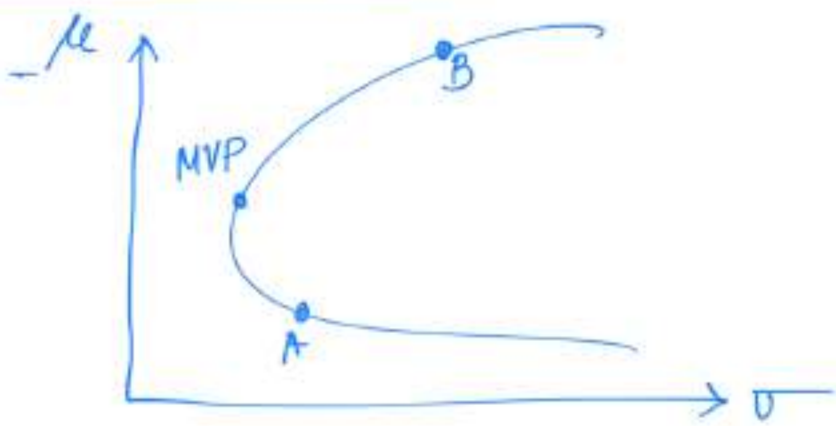
Theorem: $\sigma_1 \neq \sigma_2$.

Then

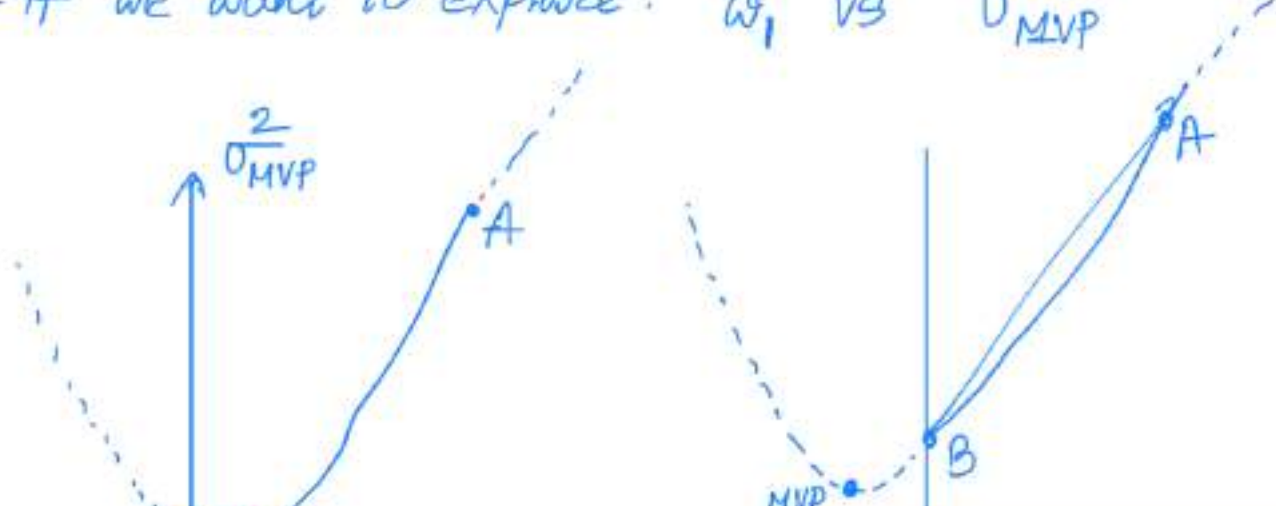
$$\mu_{MVP} = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2 - (\mu_1 + \mu_2)\rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

$$\left(\mu_{MVP} = \sum_{i=1}^{n} \omega_i \mu_i\right)$$

$$\sigma_{MVP}^2 = \frac{\sigma_1^2 \sigma_2^2 (1-\rho_{12}^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$



What if we want to explore: ω_1 Vs σ_{MVP}^2



Theorem :(shape of feasiable set) Assume $\rho_{12} \in (-1,+1)$ and $\mu_1 \neq \mu_2$ then each port V on feasiable set

$x = \sigma_V$

$y = \mu_V$

$$\Rightarrow x^2 - A^2 \left(y - \mu_{MVP}\right)^2 = \sigma_{MVP}^2 .$$

Known constant !

Def^n (Feasiable set)

The set of all portfolio which can be constructed by investing ONLY in security 1, ..., security N.

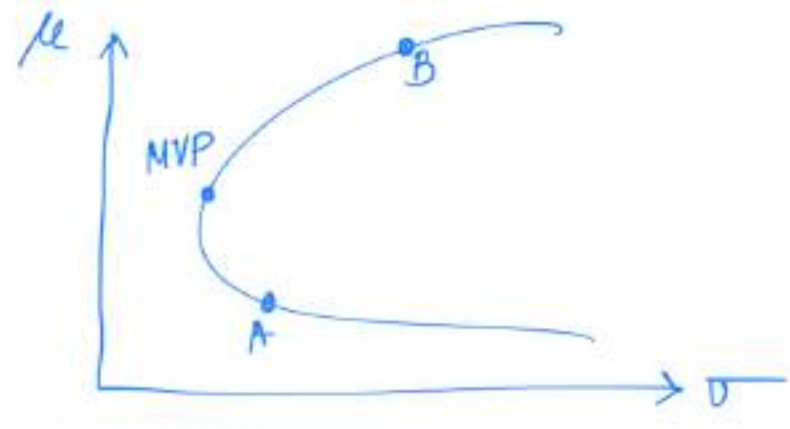
Theorem: $\sigma_1 \neq \sigma_2$.

Then

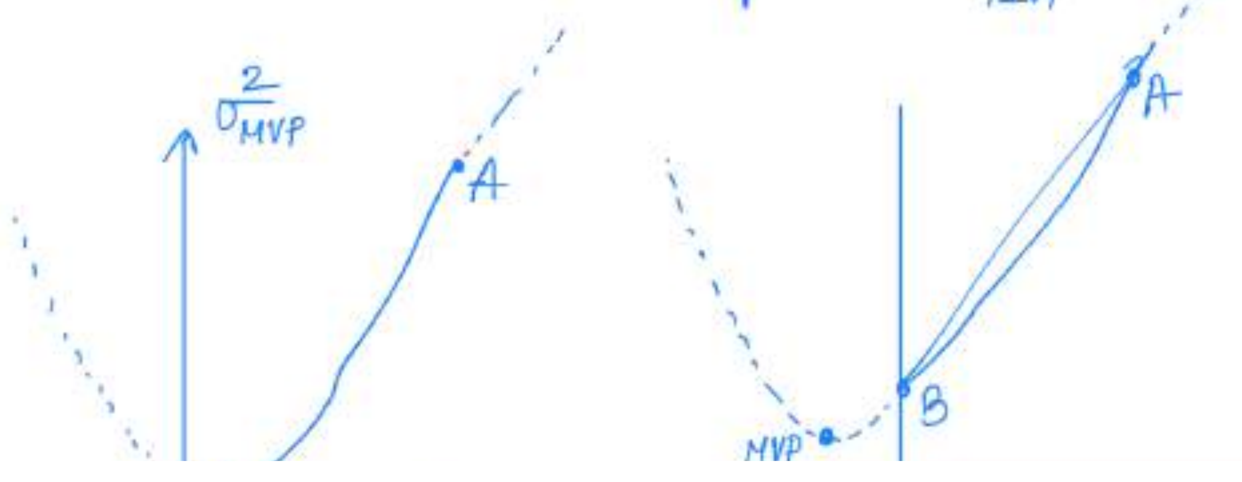
$$\mu_{MVP} = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2 - (\mu_1 + \mu_2)\rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

$$\left(\mu_{MVP} = \sum_{i=1}^{n} \omega_i \mu_i\right)$$

$$\sigma_{MVP}^2 = \frac{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$



What if we want to explore: ω_1 Vs σ_{MVP}^2



Theorem: (shape of feasiable set) Assume $\rho_{12} \in (-1, +1)$ and $\mu_1 \neq \mu_2$ then each port V on feasiable

Remark 3. *Suppose $\sigma_1 \leq \sigma_2$, then we summarize the different cases as follows:*

Cases	*Is there a portfolio with* $\sigma_V < \sigma_1 (\leq \sigma_2)$*?*	*short selling needed?*
1) $-1 \leq \rho_{12} < \frac{\sigma_1}{\sigma_2}$	*yes*	*no*
2) $\rho_{12} = \frac{\sigma_1}{\sigma_2}$	*no*	
3) $\frac{\sigma_1}{\sigma_2} < \rho_{12} \leq 1$	*yes*	*yes*

Week - 6

Math Finance — II

Notation:

N many securities.

Covariance matrix.

$$\mathbb{C}_{N\times N} = \begin{bmatrix} Cov(K_1,K_1) & \cdots & Cov(K_1,K_N) \\ \vdots & & \vdots \\ Cov(K_N,K_1) & \cdots & Cov(K_N,K_N) \end{bmatrix}$$

the $(i,j)^{th}$ element is C_{ij}

$$C_{ij} = Cov(K_i,K_j) = Cov(K_j,K_i) = C_{ji}$$

Properties

- $\mathbb{C}$ is symmetric.
- $\mathbb{C}$ is square $N\times N$ matrix.
- the diagonals are variance.
  $C_{ii} = Cov(K_i,K_i) = Var(K_i) = (\sigma_i)^2$
- $\mathbb{C}$ is always +ve semi definite

$(X C X' \geq 0 \quad \forall \; X \in \mathbb{R}^n)$

Similarly one can denote the correlation co-efficients in matrix format.

$$\sum_{N \times N} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \vdots & & & \\ \rho_{N1} & \cdots & & \rho_{NN} \end{pmatrix}$$

where ρ_{ij} := correlation coefficient of K_i and K_j

$$\left(= \frac{Cov(K_i, K_j)}{\sigma_i \, \sigma_j} = \frac{C_{ij}}{\sqrt{C_{ii} \, C_{jj}}} \right)$$

Now; a portfolio V consists of security $1, \cdots N$ can always be UNIQUElY represented as

$$\underbrace{\omega_V}_{1 \times N} = \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_N \end{pmatrix}$$

ω_i represents weights of security i

Remember: $\omega_1 + \omega_2 + \cdots \omega_N = 1$; $\omega_i \in \mathbb{R}$

Recall:

The exp. return of a portfolio $V \left(\omega_V = (\omega_1 \cdots \omega_N) \right)$ is given by:

$$K_V = \omega_1 K_1 + \omega_2 K_2 \cdots + \omega_N K_N$$

$$= \begin{pmatrix} \omega_1 & \cdots & \omega_N \end{pmatrix}_{1\times N} \begin{pmatrix} K_1 \\ \vdots \\ K_N \end{pmatrix}_{N\times 1}$$

$$= \omega_V \, K^T$$

where $K := \begin{pmatrix} K_1 & \cdots & K_N \end{pmatrix}$

$\hookrightarrow$ vector of individual returns.

Notations:

- return vector $K = \begin{pmatrix} K_1 & \cdots & K_N \end{pmatrix}_{1\times N}$
- Mean/Exp return vector $\mu = \begin{pmatrix} \mu_1 & \cdots & \mu_N \end{pmatrix}_{1\times N}$
- Cov matrix $\mathbb{C}$

$$\mathbb{C} = \begin{bmatrix} \\ \mathrm{Cov}(K_i, K_j) \\ \\ \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \sigma_{N1} & \cdots & & \sigma_N^2 \end{bmatrix}_{N\times N}$$

- Unit vector: $\mathcal{U}$

Either $\mathcal{U} := \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times N}$

or $\mathcal{U} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

depending on situation we might need row/column

$\Big)_{N\times 1}$

Theorem: The expected return and risk of the portfolio $V\left(\omega_V = (w_1 \cdots w_N)\right)$ is given by :

$$\mu_V = \sum_{i=1}^{N} \omega_i \mu_i = (\mu_1 \cdots \mu_N)\begin{pmatrix}\omega_1\\ \vdots \\ \omega_N\end{pmatrix} = \mu \omega^T \quad \left(= \omega \mu^T\right)$$

$$\sigma_V^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j \,Cov(K_i, K_j)$$

$$= \left(\omega_1 \cdots \omega_N\right)_{1\times N} \begin{pmatrix} \sigma_1^2 & & \sigma_{1N} \\ & \ddots & \\ \sigma_{N1} & & \sigma_N^2 \end{pmatrix}_{N\times N} \begin{pmatrix}\omega_1\\ \vdots \\ \omega_N\end{pmatrix}_{N\times 1}$$

$$= \left(\omega \mathbb{C} \omega^T\right)_{1\times 1}$$

Risk: $\sigma_V = \sqrt{\omega \mathbb{C} \omega^T}$

Example 1. *Given* $w_1 = 50\%$, $w_2 = -25\%$, $w_3 = 75\%$; $\mu_1 = 10\%$, $\mu_2 = 7\%$, $\mu_3 = 12\%$; $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\sigma_3 = 2$; *and* $\rho_{12} = -0.5$, $\rho_{13} = 0.5$, $\rho_{23} = 0$, *find the expected return* μ_V *and the risk* σ_V *of the portfolio.*

$$w_{V_{1\times3}} = \cancel{\left(50\% \quad -25\% \quad 75\%\right)} = \left(\frac{50}{100} \quad -\frac{25}{100} \quad \frac{75}{100}\right)$$

$$\mu_{1\times3} = \left(\mu_1 \quad \mu_2 \quad \mu_3\right) = \left(\frac{10}{100} \quad \frac{7}{100} \quad \frac{12}{100}\right)$$

$$\mathbb{C}_{3\times3} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_2^2 & \sigma_{23} \\ \cdot & \cdot & \sigma_3^2 \end{bmatrix} = \begin{bmatrix} 1^2 & -0.25 & 1 \\ -0.25 & (0.5)^2 & 0 \\ 1 & 0 & 2^2 \end{bmatrix}$$

$$\sigma_{12} = Cov(K_1 K_2) = \rho_{12}\sigma_1\sigma_2$$

$$\mu_V = \mu w^T = 0.1225$$

$$\sigma_V = \sqrt{\sigma_V^2} = \sqrt{w \, \mathbb{C} \, w^T} = \cdots \simeq 1.7984.$$

Example 2. *An investor holds a portfolio of three assets with weights* $w_1 = 0.5, \quad w_2 = -0.25, \quad w_3 = 0.75$*. The assets have expected returns and standard deviations*

$$\mu_1 = 10\%, \; \mu_2 = 7\%, \; \mu_3 = 12\%, \quad \sigma_1 = 1, \; \sigma_2 = 0.5, \; \sigma_3 = 2.$$

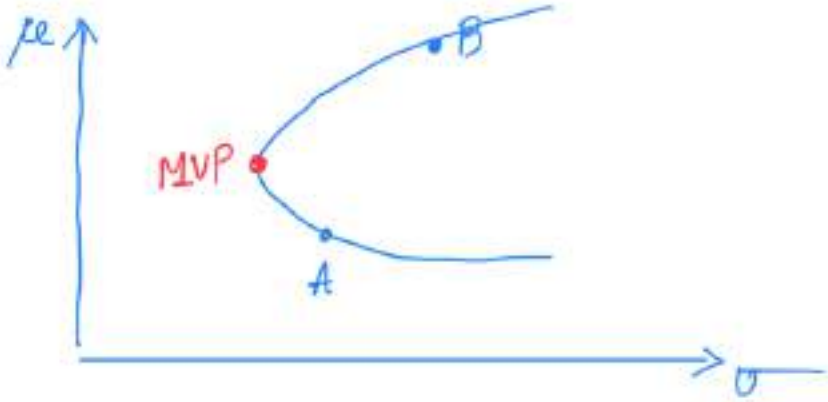
and correlations $\rho_{13} = 0.5$, $\rho_{23} = 0$, *while* ρ_{12} *is unknown. The portfolio standard deviation is observed to be* $\sigma_V = 1.824$.

(a) *Find the covariance* $\mathrm{Cov}(K_1, K_2)$ *and correlation coefficient* $\rho_{1,2}$.

(b) *Compute the expected return* μ_V *of the portfolio.*

Min Var Portfolio

For N=2



as long as $|\rho_{AB}| \neq 1$

From last class:

$$w_1^{min} = \frac{\sigma_2^2 - \underbrace{\rho_{12}\sigma_1\sigma_2}_{=\sigma_{12}}}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} = \frac{\sigma_{22} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}$$

$$w_2^{min} = 1 - w_1^{min}$$

$$= \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} = \frac{\sigma_{11} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}$$

(Remember $\sigma_{12} = \sigma_{21}$)

① $\det(\mathbb{C})\, \underset{1\times N}{u}\, \mathbb{C}^{-1}_{N\times N}$

② $(\det \mathbb{C})\, u \mathbb{C}^{-1} u^T$

$$\mathbb{C} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$\sigma_{ij} := Cov(k_i, k_j)$

$\sigma_{ij} = \sigma_{ji}$

$$\mathbb{C}^{-1} = \frac{1}{(\det \mathbb{C})} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}$$

$$① : \det(\mathbb{C})\, u\mathbb{C}^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{22} - \sigma_{12} & \sigma_{11} - \sigma_{12} \end{bmatrix}$$

$$② : \det(\mathbb{C})\, u\mathbb{C}^{-1} u^T = \begin{bmatrix} \sigma_{22} - \sigma_{12} & \sigma_{11} - \sigma_{12} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \sigma_{11} + \sigma_{22} - 2\sigma_{12}$$

$$\frac{①}{②} = \frac{\begin{bmatrix} \sigma_{22} - \sigma_{12} & \sigma_{11} - \sigma_{12} \end{bmatrix}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}$$

$$= \begin{bmatrix} \dfrac{\sigma_{22} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}} & \dfrac{\sigma_{11} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}} \end{bmatrix}$$

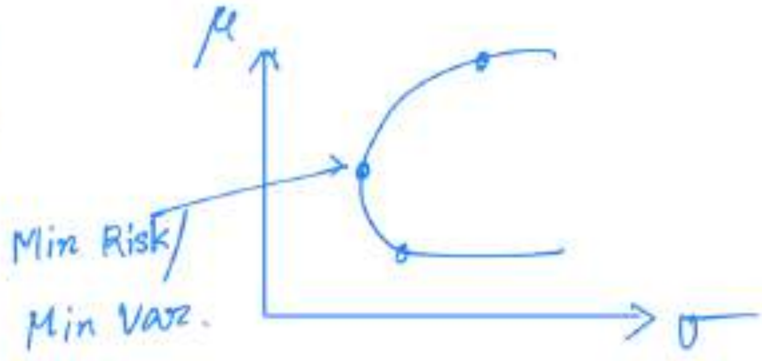
$$= \begin{bmatrix} \omega_1^{min} & \omega_2^{min} \end{bmatrix} = \omega_{MVP} \left(= \frac{①}{②} = \frac{u\mathbb{C}^{-1}}{u\mathbb{C}^{-1}u^T} \right)$$

Theorem: (N security).

Assume $\det \mathbb{C} \neq 0$. Then the MVP weights are given by:

Imp
$$\boxed{\omega_{MVP} = \frac{u\mathbb{C}^{-1}}{u\mathbb{C}^{-1}u^T}}$$

Proof:



We want to min Risk

$\Updownarrow$

min var.

Solve:

$$\min_{\omega_v} \; \omega_v \, \mathbb{C} \, \omega_v^T$$

$$\text{st} \quad \sum_{i=1}^{N} \omega_i = 1 \iff u\omega_v^T = 1$$

The lagrange problem:

$$\mathcal{L}(\omega_v, \lambda) = \omega_v \, \mathbb{C} \, \omega_v^T - \lambda \left(u\omega_v^T - 1 \right)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial \omega_v} = 0 \iff 2\omega_v \mathbb{C} - \lambda u = 0$$

$$\iff \omega_v = \frac{\lambda u \mathbb{C}^{-1}}{2} \quad \text{— } (*)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff u\omega_v^T = 1$$

$\hookrightarrow$ replacing $(*)$ we get

$$\frac{\lambda}{2} \, u C^{-1} u^T = 1$$

$$\Leftrightarrow \lambda = \frac{2}{u C^{-1} u^T} \quad \text{———} (**)$$

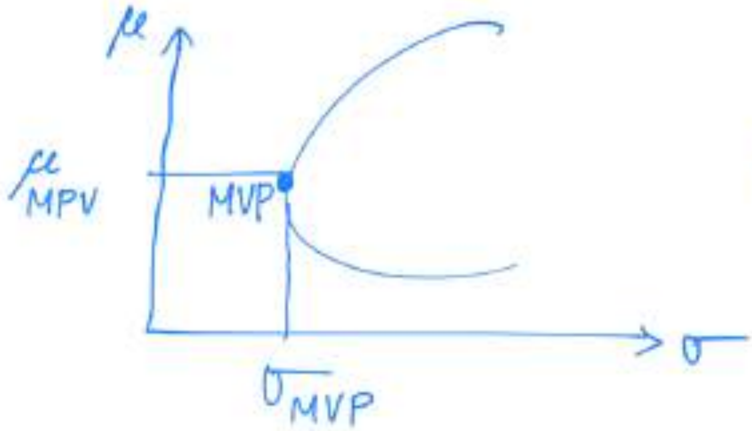
replacing λ in ($*$) we get

$$w^* = w_{MVP} = \frac{u C^{-1}}{u C^{-1} u^T}$$

Sometimes w_{MVP} is written as.

$$w_{MVP} = \frac{C^{-1} u^T}{u C^{-1} u^T}$$ ← column vector instead of row vec.

The two followup Qn.



w_{MVP}

Imp

Theorem:

$$\boxed{\mu_{MVP} = \frac{u C^{-1} \mu^T}{u C^{-1} u^T}}$$

$$\boxed{\sigma_{MVP} = \sqrt{\sigma^2_{MVP}} = \sqrt{\frac{1}{u C^{-1} u^T}}}$$

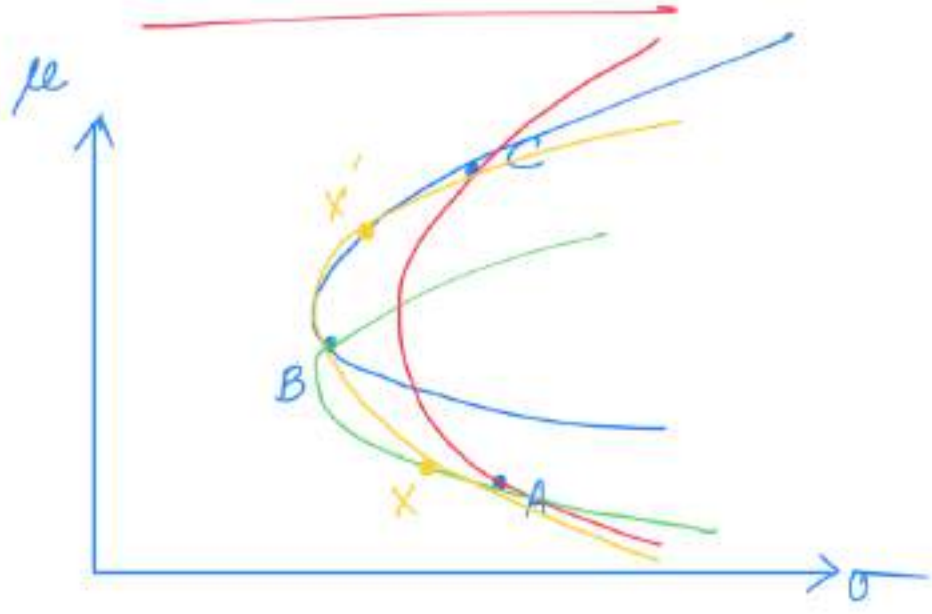
Proof: $\sigma^2_{MVP} = \omega_{MVP}\, \mathbb{C}\, \omega_{MVP}^T$

$$= \left(\frac{u\mathbb{C}^{-1}}{u\mathbb{C}^{-1}u^T}\right) \mathbb{C} \left(\frac{u\mathbb{C}^{-1}}{u\mathbb{C}^{-1}u^T}\right)^T$$

$$= \frac{u\cancel{\mathbb{C}^{-1}}\,\cancel{\mathbb{C}}\,\underbrace{(\mathbb{C}^{-1})^T}_{\mathbb{C}^{-1}}u^T}{(u\mathbb{C}^{-1}u^T)^2}$$

$$= \frac{\cancel{u\mathbb{C}^{-1}u^T}}{(u\mathbb{C}^{-1}u^T)^{\cancel{2}}} = \frac{1}{u\mathbb{C}^{-1}u^T}$$

Feasiable set



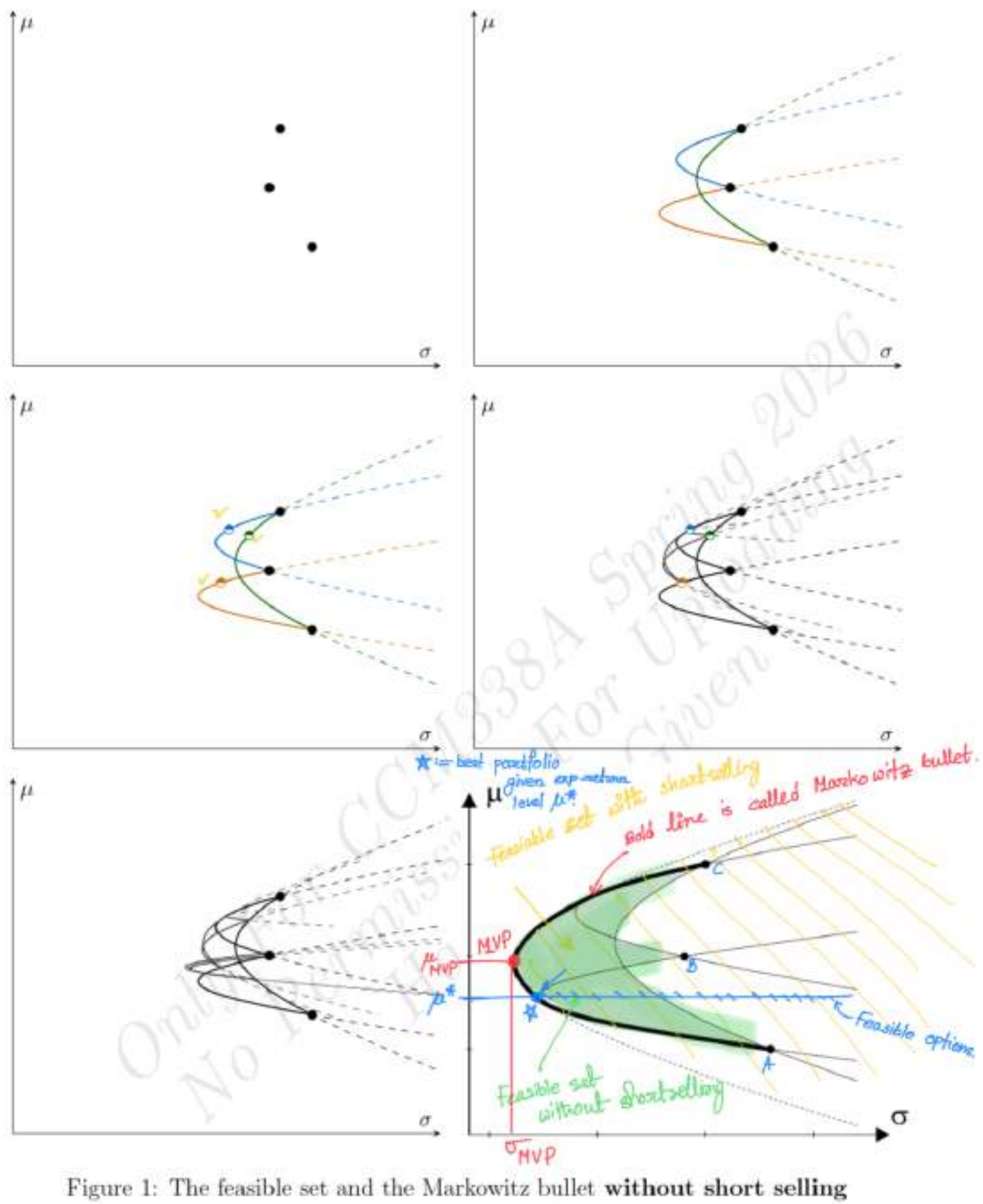


Figure 1: The feasible set and the Markowitz bullet **without short selling**

Assumption :

We need atleast 3 securities above.

+ all securities has different exp return ($\mu_1 \neq \mu_2 \neq \mu_3$)

+ $\rho_{AB}, \rho_{BC}, \rho_{CA} \notin \{-1, +1\}$

Under the assumption the feasible set will always look like above.

Note: The solid line is called Markowitz bullet. this is also the left most points of feasiable set.

Imp

Note: Markowitz bullet is also known as minimum variance Line (MVL).

Defⁿ: (MVL)

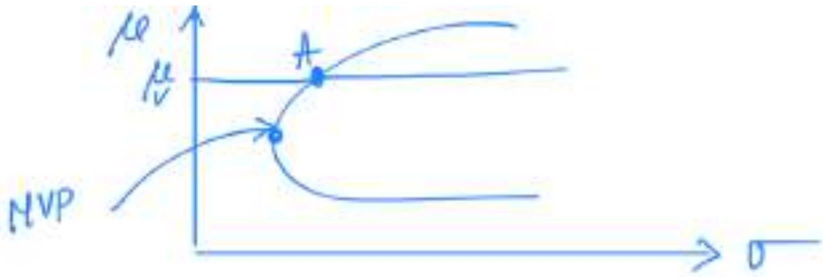
set of all portfolio with min risk given the expected return μ^*.

Theorem 2.4 (Weights of portfolios on the minimal variance line). *Assume that* $\det \mathbb{C} \neq 0$. *Fix a level of expected return* μ_V. *Then a portfolio has the smallest risk among all portfolios with expected return* μ_V *if and only if it has the weights*

(Assump) $\mu_V \geq \mu_{MVP}$

$$w_{\mu_V}^{\min} = \frac{\begin{vmatrix} 1 & u\mathbb{C}^{-1}\mu^\top \\ \mu_V & \mu\mathbb{C}^{-1}\mu^\top \end{vmatrix} u\mathbb{C}^{-1} + \begin{vmatrix} u\mathbb{C}^{-1}u^\top & 1 \\ \mu\mathbb{C}^{-1}u^\top & \mu_V \end{vmatrix} \mu\mathbb{C}^{-1}}{\begin{vmatrix} u\mathbb{C}^{-1}u^\top & u\mathbb{C}^{-1}\mu^\top \\ \mu\mathbb{C}^{-1}u^\top & \mu\mathbb{C}^{-1}\mu^\top \end{vmatrix}}, \tag{6}$$

where we have used the usual notation for the determinant of a 2×2 *matrix.*



Proof:

Solve:

$$\min_{w} \; w \mathbb{C} w^{\top}$$

$$\text{st} \quad w u^{\top} = 1$$

$$\mu w^{\top} = \mu_V$$

Similarly to a previous case, we will use the Lagrange multipliers method. To this end, define

$$\mathcal{L}(w, \lambda) := w\mathbb{C}w^{\top} - \lambda_1(uw^{\top} - 1) - \lambda_2(\mu w^{\top} - \mu_V).$$

Then,

$$\nabla_w \mathcal{L} = 0 \Leftrightarrow 2w\mathbb{C} - \lambda_1 u - \lambda_2 \mu = 0 \Leftrightarrow w = \frac{1}{2}(\lambda_1 u + \lambda_2 \mu)\mathbb{C}^{-1} \tag{7}$$

which, in conjunction with the constraints ($uw^{\top} = 1$ and $\mu w^{\top} = \mu_V$), leads to the system with unknowns only involving λ:

$$\begin{cases} \lambda_1 u\mathbb{C}^{-1}u^{\top} + \lambda_2 u\mathbb{C}^{-1}\mu^{\top} = 2 \\ \lambda_1 \mu\mathbb{C}^{-1}u^{\top} + \lambda_2 \mu\mathbb{C}^{-1}\mu^{\top} = 2\mu_V \end{cases}, \tag{8}$$

whose solution can be immediately determined, *e.g.*, by Cramer's rule. Then, we derive immediately $w_{\mu_V}^{\min}$ by substituting in (7). $\square$

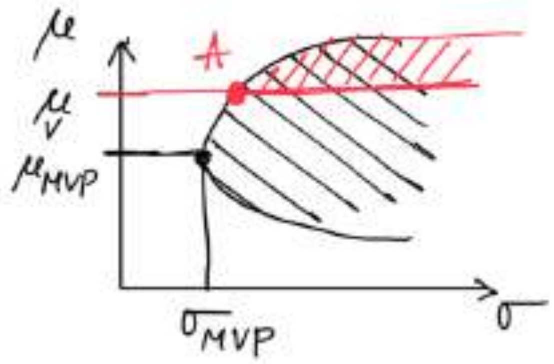
Week – 7

Math-Finance – II

Reminder: Please submit "Tutorial-5 suggestion" by Friday.

Recap:

The last theorem from last class:



Theorem (Weights of portfolio on the MVL)

weights of A

$$w^{min}_{\mu_V} = \frac{\begin{vmatrix} 1 & uC^{-1}\mu^T \\ \mu_V & \mu C^{-1}\mu^T \end{vmatrix} C^{-1}u^T + \begin{vmatrix} uC^{-1}u^T & 1 \\ \mu C^{-1}u^T & \mu_V \end{vmatrix} C^{-1}\mu^T}{\begin{vmatrix} uC^{-1}u^T & uC^{-1}\mu^T \\ \mu C^{-1}u^T & \mu C^{-1}\mu^T \end{vmatrix}}$$

— $A \in \mathbb{R}$
— $B \in \mathbb{R}$
— $C \in \mathbb{R}$
} — scalar.

$\boxed{\text{Note: } \mu C^{-1}u^T = uC^{-1}\mu^T}$

Assump: $A > 0$ and $C > 0$.

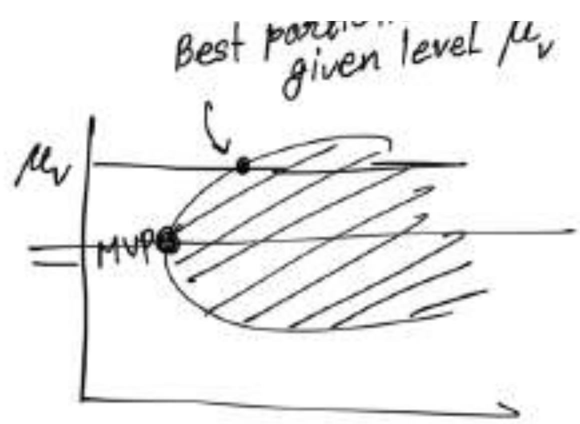
$$w^{min}_{\mu_V} = \frac{\begin{vmatrix} 1 & B \\ \mu_V & C \end{vmatrix} C^{-1}u^T + \begin{vmatrix} A & 1 \\ B & \mu_V \end{vmatrix} C^{-1}\mu^T}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}}$$

$$= \frac{(C - \mu_V B)\, C^{-1}u^T + (A\mu_V - B)\, C^{-1}\mu^T}{AC - B^2}$$

folio

To do:

"Two fund theorem"



☒ Finding MVP from $\omega_{\mu_v}^{min}$.

chose $\mu_v := \frac{B}{A}$

$$\omega_{\mu_v}^{min} = \frac{(C - \mu_v B)\, \mathbb{C}^{-1} u^T + \cancel{(A\mu_v - B)}^{\,0}\, \mathbb{C}^{-1} \mu^T}{AC - B^2}$$

$$= \frac{\left(C - \frac{B^2}{A}\right) \mathbb{C}^{-1} u^T}{AC - B^2}$$

$$= \frac{\cancel{(AC - B^2)}\, \mathbb{C}^{-1} u^T}{\cancel{(AC - B^2)}\, A} = \frac{\mathbb{C}^{-1} u^T}{A}$$

$$= \boxed{\frac{\mathbb{C}^{-1} u^T}{u\, \mathbb{C}^{-1} u^T} = \omega_{MVP}}$$

Finding second efficient portfolio:

Chose $\mu_v := C/B$

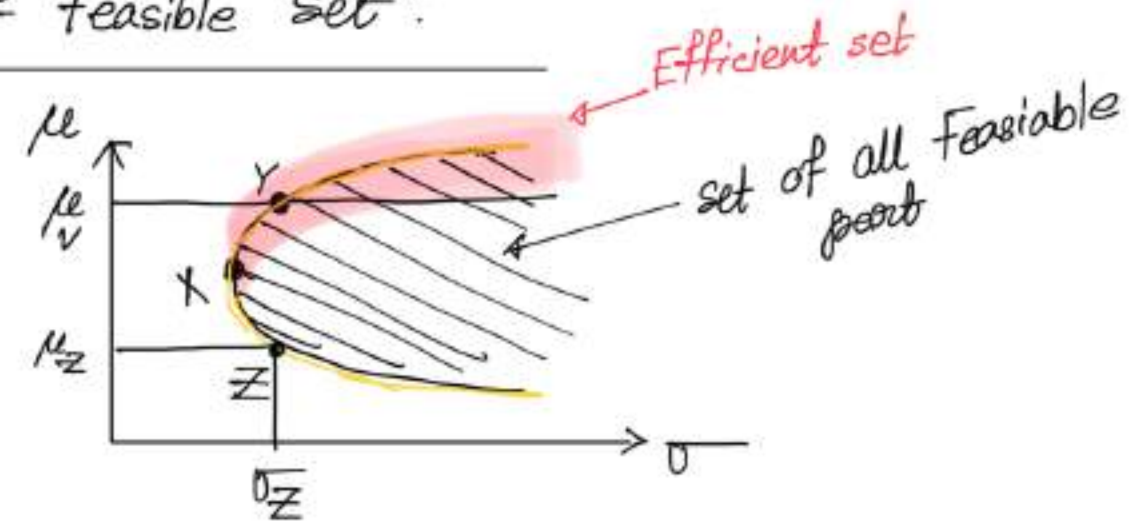
$$\omega_{\mu_v}^{min} = \frac{\cancel{(C - \mu_v B)}^{\,0}\, \mathbb{C}^{-1} u^T + (A\mu_v - B)\, \mathbb{C}^{-1} \mu^T}{AC - B^2}$$

$$= \frac{\left(\frac{AC}{B} - B\right) \mathbb{C}^{-1} \mu^T}{\dots} \qquad \mathbb{C}^{-1} \mu^T$$

$$= \frac{(\ldots)}{(AC-B^2)} = \frac{\ldots}{B}$$

$$\boxed{= \frac{\mathbb{C}^{-1}\mu^T}{\mu\,\mathbb{C}^{-1}u^T} = W^{\text{second eff port}} \quad \mu\,\mathbb{C}^{-1}u^T = u\,\mathbb{C}^{-1}\mu^T}$$

Prop. of Feasible Set:



→ Any security can be uniquely represented in $\sigma-\mu$ plane.

→ X := Best portfolio i.e. the port with Min risk
= MVP.

→ We learnt how to construct the best port (Y) given exp return μ_Y.

→ —— The yellow line := Markowitz Bullet
also known as Min Variance Line (MVL)

→ Any portfolio which are efficient has to be on MVL, but not all points in MVL are efficient portfolios.

Theorem 2.5 (Risk associated to the best portfolio given expected return μ_V). *Assume that* $\det \mathbb{C} \neq 0$. *Fix a level of expected return* μ_V. *Then the portfolio that has the smallest risk among all portfolios with expected return* μ_V *has risk/variance as follows.*

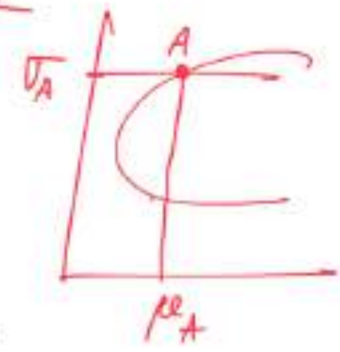
$$\sigma^2_{w^{\min}_{\mu_V}} = \sigma^2_{\mathrm{MVP}} + \frac{(\mu_V - \mu_{\mathrm{MVP}})^2}{\psi \sigma^2_{\mathrm{MVP}}}, \quad \textit{with},$$

const

$$\psi = AC - B^2 = u^\top \mathbb{C}^{-1} u \mu^\top \mathbb{C}^{-1} \mu - (u^\top \mathbb{C}^{-1} \mu)^2.$$

Proof. We will skip the proof, as this is computationally heavy. □

$$\sigma_A^2 = \sigma_{MVP}^2 + \underbrace{Cont \times (\mu_A - \mu_{MVP})^2}_{\text{conpensation for extra exp return.}}$$



Additionally:

Lemma:

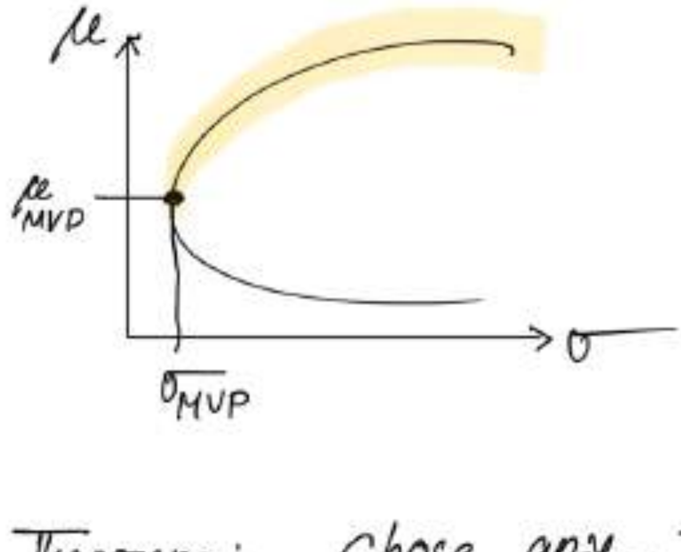
$$\omega_A = Const^{①} \mu_A + Const^{②}$$

↳ weights of best portfolio at exp return level μ_V depends linearly on μ_V.

Note: The above lemma is very useful in practic as once the cont① and const② are estimated it is very easy to find $\omega_{\mu_V}^{min}$.

The portfolio $\omega_{\mu_V}^{min}$ is known as "target return portfolio".

Two fund theorem



— set of all efficient port.

Theorem: Chose any two arbitarily distict portfolio on the MVL, say V_1^{MVL} and V_2^{MVL}, and these has weights denoted as ω_1 and ω_2 and exp return μ_1 and μ_2 ($\mu_1 \neq \mu_2$).

Then a portfolio V is on MVL iff it's weight vector (w) can be written as a afine combination of w_1 and w_2.

Mathematically: $\boxed{w = \alpha w_1 + (1-\alpha) w_2}$ for some $\alpha \in \mathbb{R}$.

$\uparrow$ weight of V

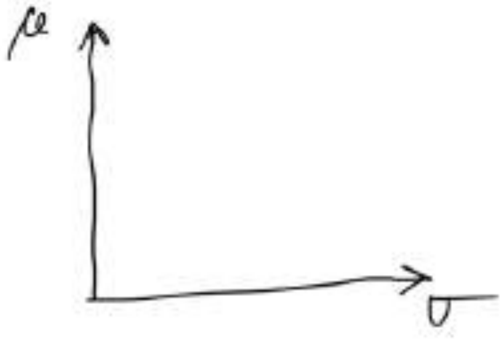
In other words:

One can construct two efficient funds/portfolios, st any other efficient portfolio is jus a combinition of these two funds.

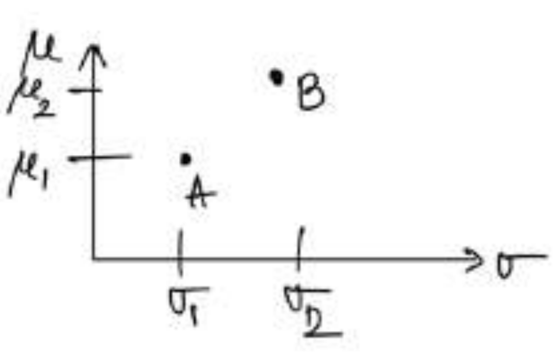
Two distinct portfolio:

$V_1 = (\sigma_1, \mu_1)$

$V_2 = (\sigma_2, \mu_2)$.



- If $\mu_1 = \mu_2 \longrightarrow$ the one with lower risk.
- If $\sigma_1 = \sigma_2 \longrightarrow$ the one with better exp return.
- If $\sigma_1 \leq \sigma_2$ and $\mu_1 \geq \mu_2 \longrightarrow$ the portf (σ_1, μ_1)
- If $\sigma_1 \leq \sigma_2$ and $\mu_1 \leq \mu_2 \longrightarrow$ No obvious best
  $\searrow$ depens on the level of risk-aversion.

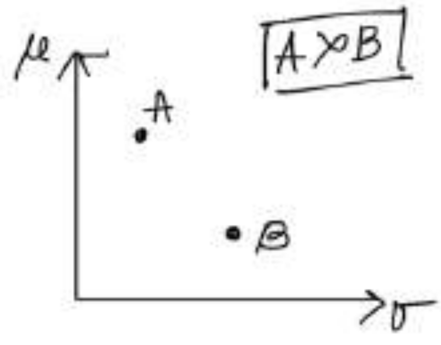


Efficient Frontier

Def^n: (Dominating Portfolio)

We say a portfolio (σ_1, μ_1) dominates another portfolio (σ_2, μ_2) if

$$\mu_1 \geq \mu_2 \text{ and } \sigma_1 \leq \sigma_2$$

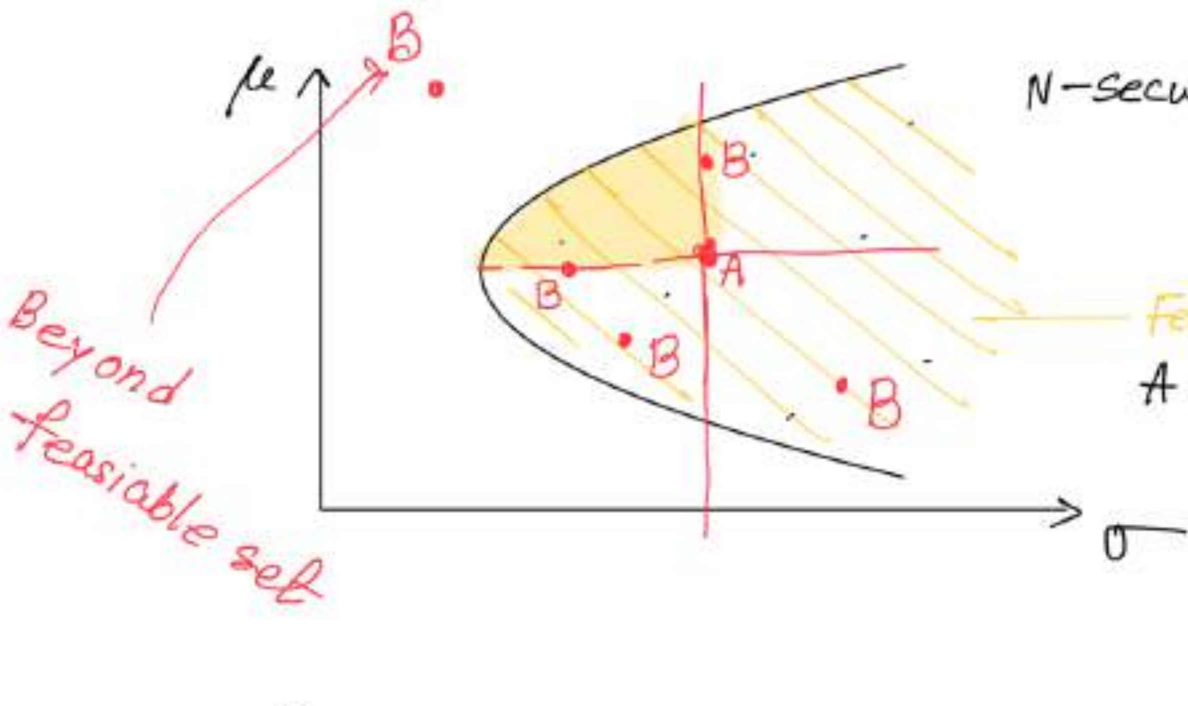


"Best Portfolio" — these are nothing but Efficient portfolio

Def^n (Efficient port)

A portfolio is called efficient port if no other portfolio dominates ~~this~~ portfolio.

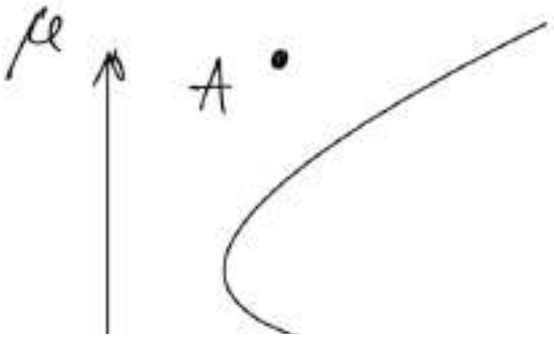
Notation: set of all efficient portfolio form <u>efficient frontier</u> (EF).



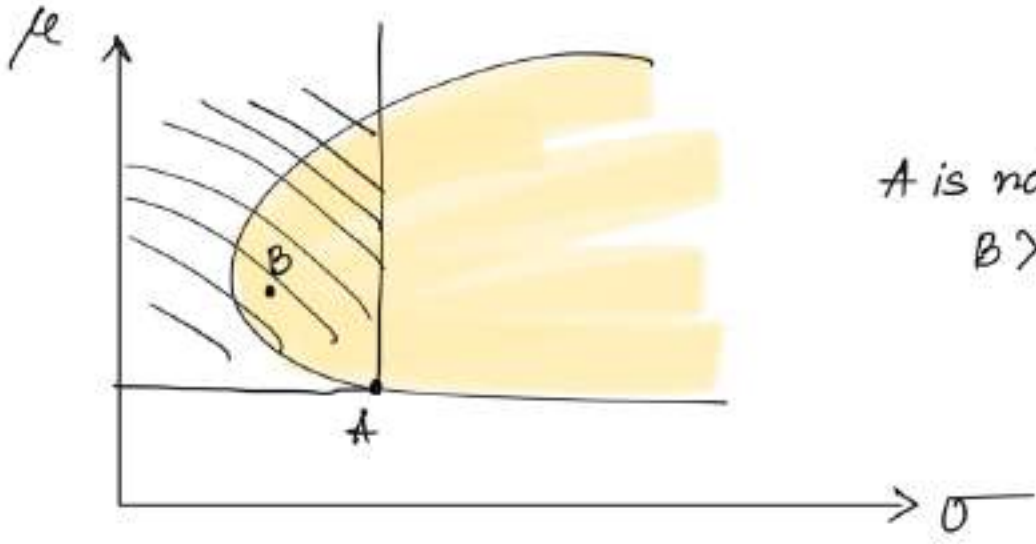
N-securities.

Feasiable set

A is not on EF as we can find port that dominates A.

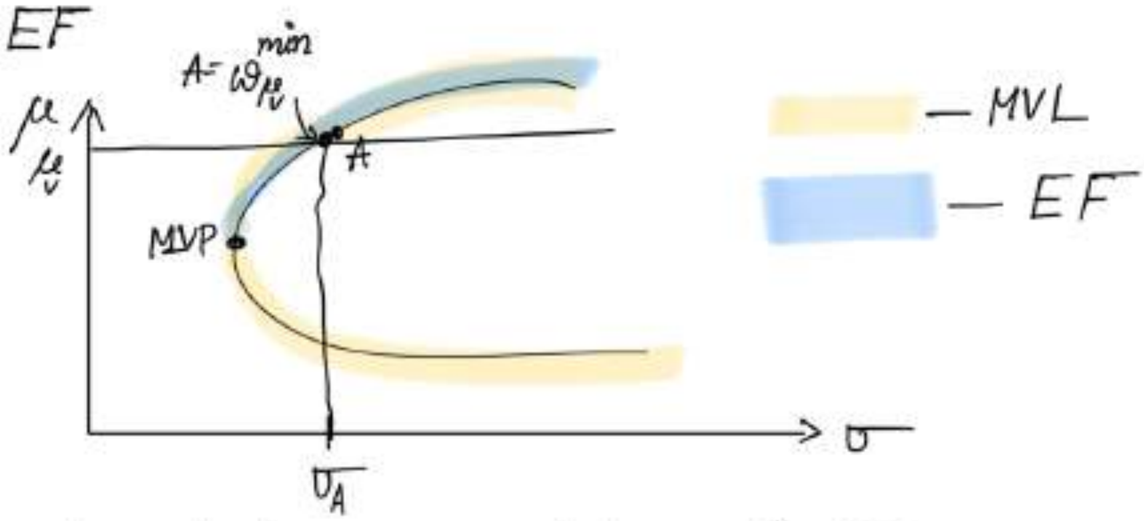


A is not on EF as it is beyond feasiable set



A is not on EF as $B \succ A$.

So: The EF



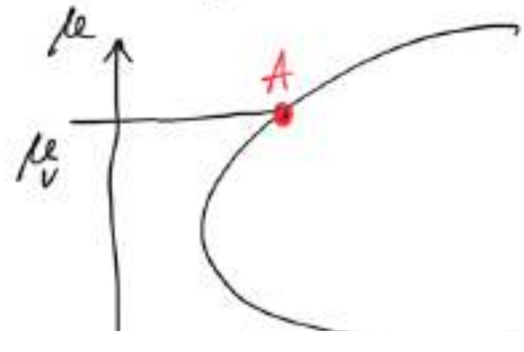
What is the mathematical representation of EF:

$$\left\{ \left(\sigma_{\omega^{min}_{\mu_V}}, \mu_V\right) \middle| \ \mu_V \geq \mu_{MVP} \right\}$$

Qn: Say security A dominates security B.
Do we get rid of security B from optimal port?

→ No (Diversification)

Qn: I fix the level of exp return μ_V.



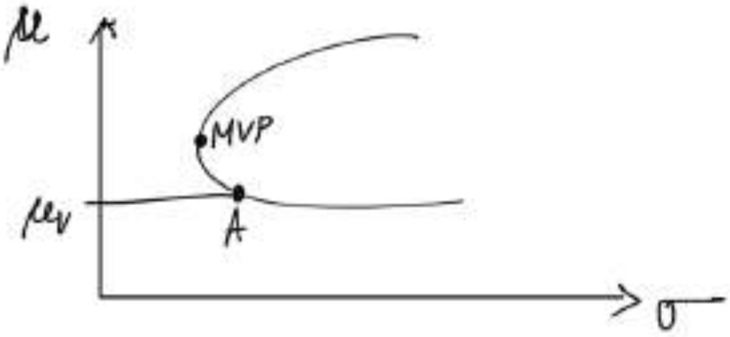
Case-1
My exp return $= \mu_V \Rightarrow A$

Case-II

case

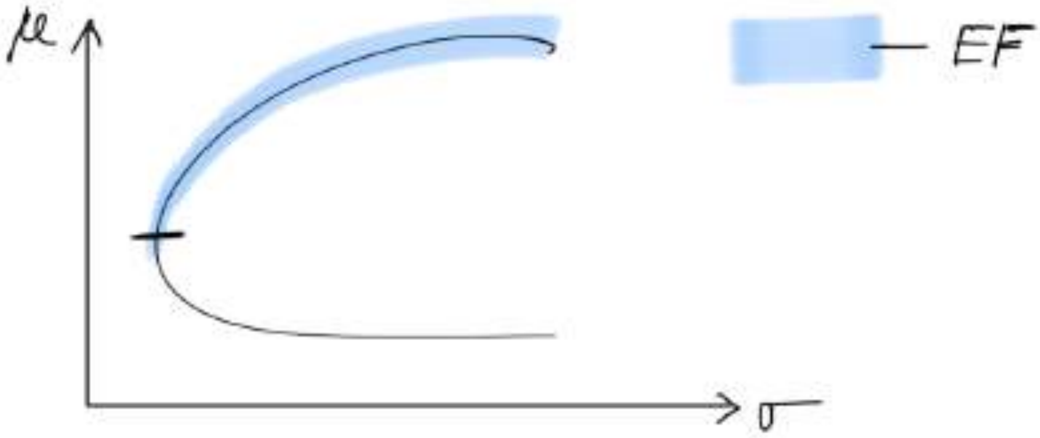
My exp return is at least μ_V 

$\Longrightarrow A$



<u>Lemma:</u> The EF is the set of all portfolios whose weights ω_{EF} are given by:

$$\omega_{EF} = \text{Cont}^{①} \mu_{EF} + \text{Cont}^{②}$$



<u>Goal!</u> I want to chose the best possiable port from EF.

<u>Recall:</u> In utility theory we had Quadratic utility fn:

$$U(\omega) = \omega - \frac{b}{2}\omega^2 \quad ; \; b>0.$$

We were max $\mathbb{E}(U(W))$.

Now we will define Performance criteria of a port.

$$:= \mathbb{E}(R_V) - \frac{\gamma}{2} \operatorname{Var}[R_V]$$

$$= \omega_V^T \mu - \frac{\gamma}{2} \omega_V^T \mathbb{C} \omega_V .$$

Once we fix the level (of risk-aversion) γ the we can construct a portfolio which is on EF as well as gives the highest Performance criteria.

Theorem 3.1. *The weights associated to the portfolio with the best portfolio performance criteria is given by(No need to remember the formulas):*

$$w_{\text{best-performence}} = \frac{\mathbb{C}^{-1}u}{u^\top \mathbb{C}^{-1}u} + \frac{1}{\gamma}\left(\frac{u^\top \mathbb{C}^{-1}u\mathbb{C}^{-1}\mu - \mu^\top \mathbb{C}^{-1}u\mathbb{C}^{-1}u}{u^\top \mathbb{C}^{-1}u}\right).$$

Furthermore, the expected return and the variance of this optimal portfolio if given by

$$\mu_{\text{best-performence}} = \frac{u^\top \mathbb{C}^{-1}\mu}{u^\top \mathbb{C}^{-1}u} + \frac{1}{\gamma}\left(\frac{\mu^\top \mathbb{C}^{-1}\mu u^\top \mathbb{C}^{-1}u - (u^\top \mathbb{C}^{-1}\mu)^2}{u^\top \mathbb{C}^{-1}u}\right)$$

$$= \mu_{MVP} + \frac{1}{\gamma}\psi\sigma^2_{MVP}.$$

$$\sigma^2_{\text{best-performence}} = \sigma^2_{MVP} + \frac{1}{\gamma^2}\left(\frac{\mu^\top \mathbb{C}^{-1}\mu u^\top \mathbb{C}^{-1}u - (u^\top \mathbb{C}^{-1}\mu)^2}{u^\top \mathbb{C}^{-1}u}\right)$$

$$= \sigma^2_{MVP} + \frac{1}{\gamma^2}\psi\sigma^2_{MVP}.$$

Where $\psi := \mu^\top \mathbb{C}^{-1}\mu u^\top \mathbb{C}^{-1}u - (u^\top \mathbb{C}^{-1}\mu)^2$ is a scaler value. Note that this ψ is the same constant which appears in Theorem 2.5. This is a consequence of $\psi = \psi^T$ (since ψ is scaler) and $\mathbb{C}^{-1} = (\mathbb{C}^{-1})^T$ (Since $\mathbb{C}$ is symmetric).

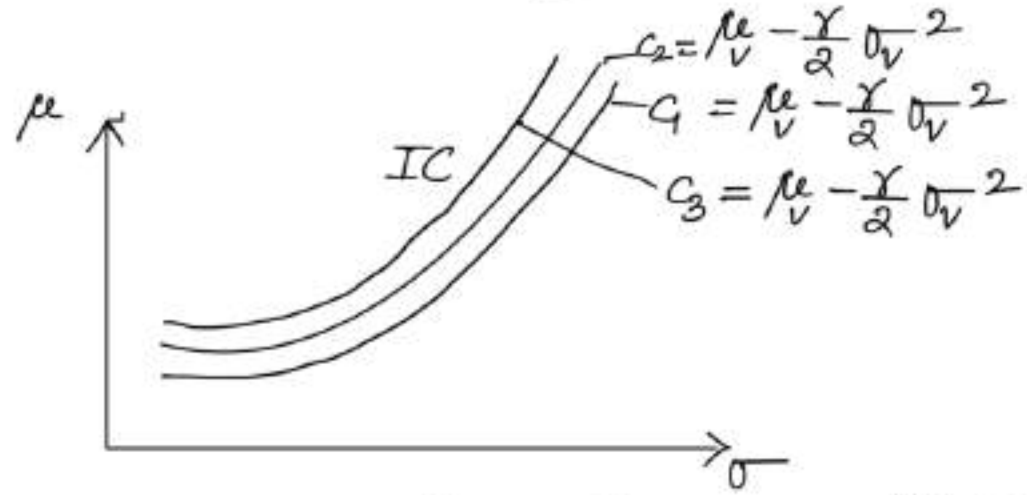
Proof. Proof using the Lagrange multiplier. □

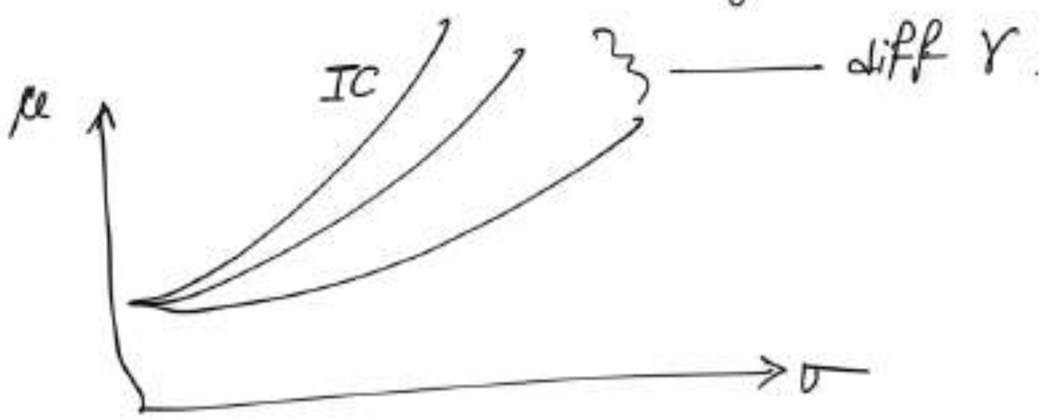
Indifference Curve:

Def^n: An indifference cure is a collection of portfolios which an investor is indifferent to.

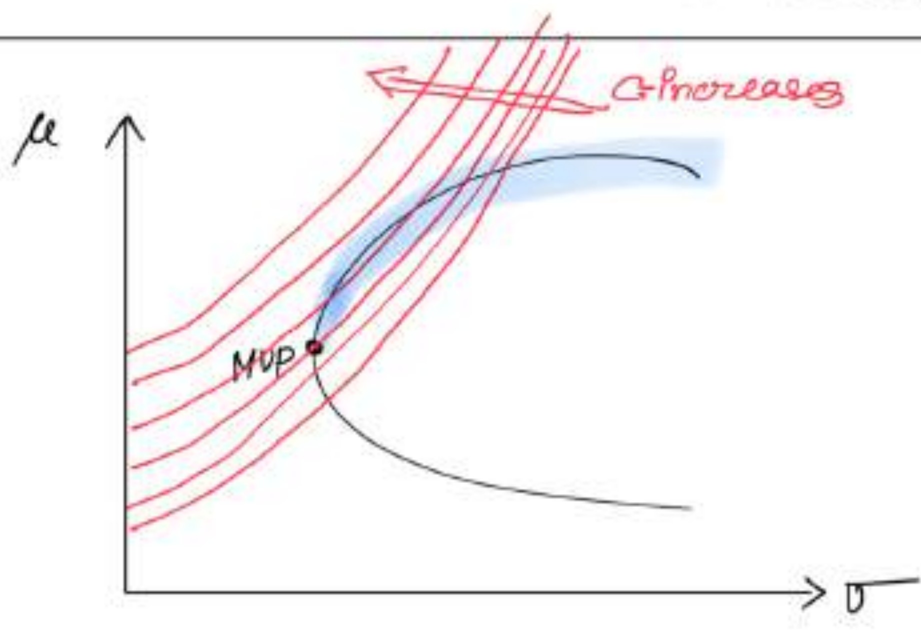
i.e. for all the portfolios;

$$\mu_V - \frac{\gamma}{2}\sigma_V^2 = C_0$$

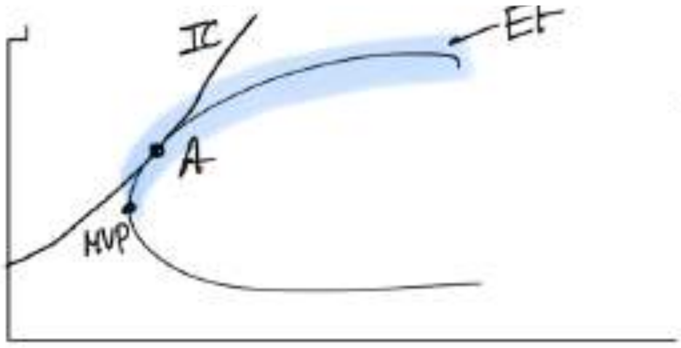




Which portfolio on EF gives me best performance criteria.



$\Downarrow$



A → Max performance criteria on EF.

Week — 8

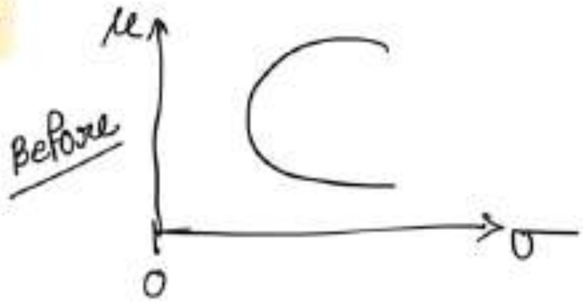
Math - Finance — II

Remainder: Please submit "Tutorial-7 suggession" by Friday.

$\boxed{\text{Class Test: } 11^{th} \text{ March (Wed)}}$

Recap: Until week 7:

MVP, EF, Feasiable set, Exp return and risk of a portfolio for N — Risky security.



Add a Risk-free asset

Framework: N- risky asset + 1 Risk free.

Notations: $R / R_{rf} / R_f / \mu_{RF}$ — of risk free asset.

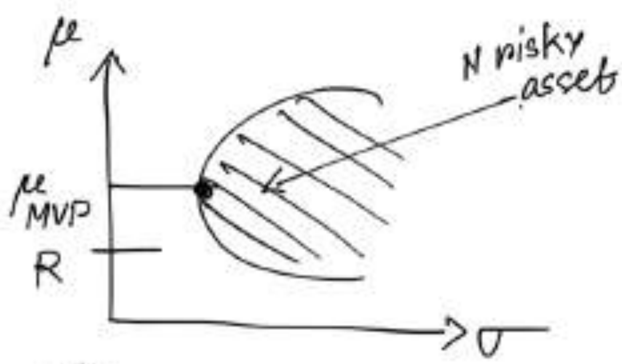
R is not a Random variable.

$$\boxed{\underbrace{\mathbb{E}(K_R)}_{\mu_R} = K_R = R}$$

Exp rate of return = Rate of return!

$\boxed{\text{Assump: } R < \mu_{MVP}}$

↳ Bond will not give better exp return than Risky asset* (best).



Step – 1

What is the feasible set when we have 1 Risky asset (V) + 1 RF (R)

Take a new portfolio **P** with $\omega = (\omega_f, \omega_1)$ with

ω_f representing weight of RF

ω_1 ———————— Risky asset V.

$\boxed{\text{Note : } \omega_f + \omega_1 = 1}$

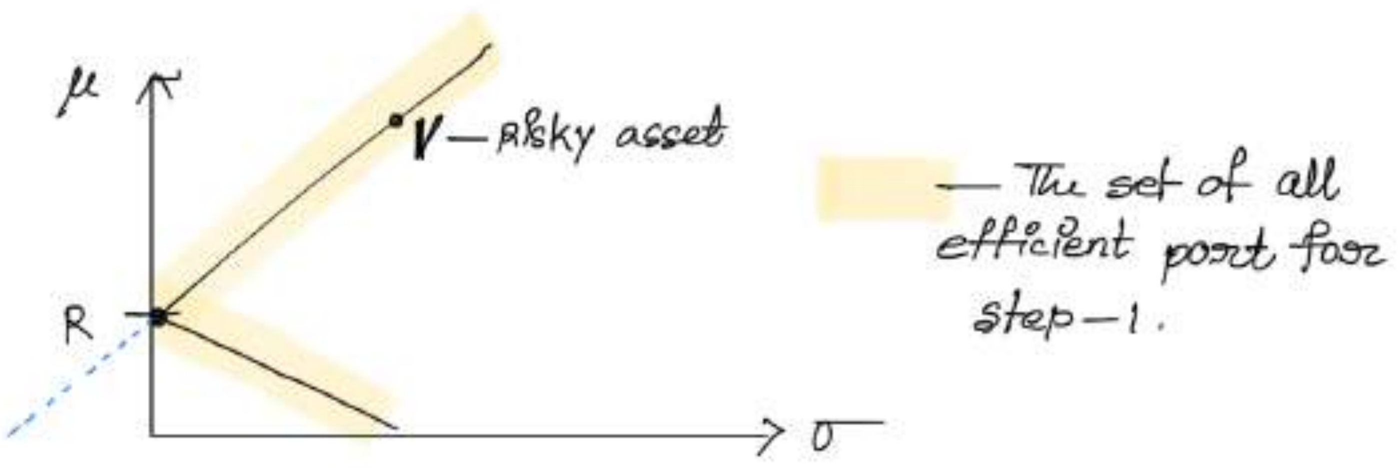
$$\mu_P = \omega_f \mu_f + \omega_1 \mu_1 = (1-\omega_1)\mu_f + \omega_1 \mu_1$$

$$\sigma_P^2 = \omega_f^2 \underbrace{\sigma_f^2}_{0} + \omega_1^2 \sigma_1^2 + 2\omega_f \omega_1 \underbrace{\mathrm{Cov}(K_f, K_1)}_{0\ (HW)}$$

$$= \omega_1^2 \sigma_1^2$$

$$\Longleftrightarrow \sigma_P = |\omega_1 \sigma_1|$$

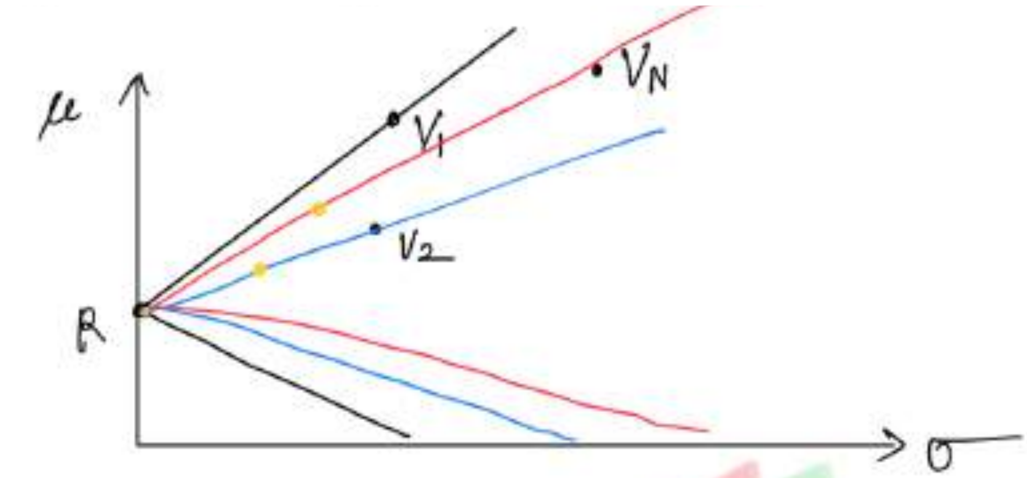
$\left.\begin{array}{l} \mu_P \text{ is linear on } \omega_1 \\ \sigma_P \text{ is also "linear" on } \omega_1 \end{array}\right\}$ — σ_P is "linear" on μ_P.

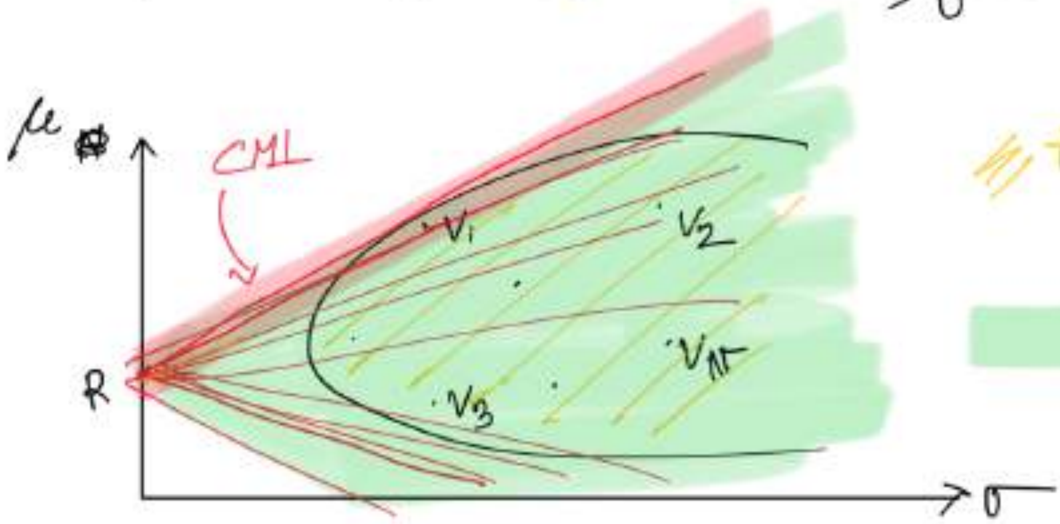


Step - 2 N – Risky asset + 1 RF. (One should not have more than 1 RF in a market

$(R, V_1, V_2 \cdots V_N)$

→ we remove the one with lower interest



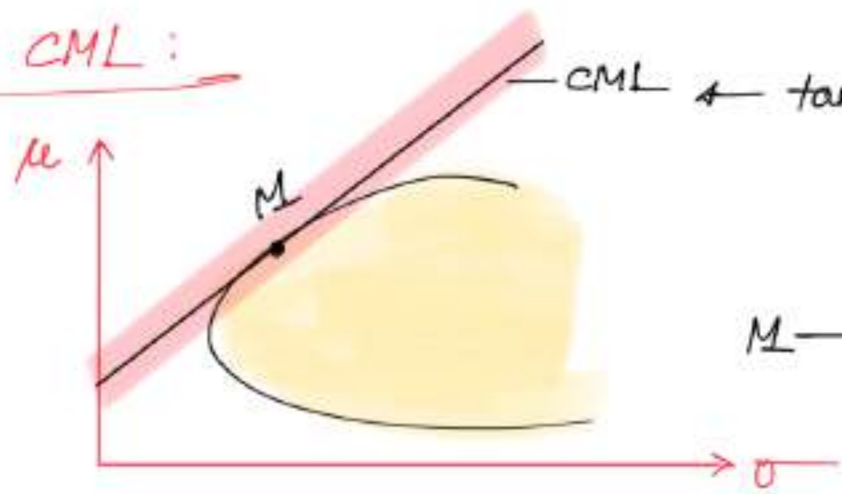


Feasible set with just N-Risky asset

— feasible port with N-risky + 1 RF.

— set of all efficient port (under RF) (CML)

what is CML :



CML ← tangent to the old feasible set (No-RF)

M — Market portfolio.

M is the only efficient port which can be constructed just from the N-risky asset.

Definition 10 (CML, Market Portfolio). *The half-line that starts at the risk-free asset and runs through the market portfolio M is called the* Capital Market Line *(CML). The portfolio M corresponding to the tangency point (σ_M, μ_M) is called the* market portfolio.

Theorem: (One fund theorem).

Assume (N-risky + 1 RF). Then there exists a unique single fund (Market port M) st any efficient port single fund (

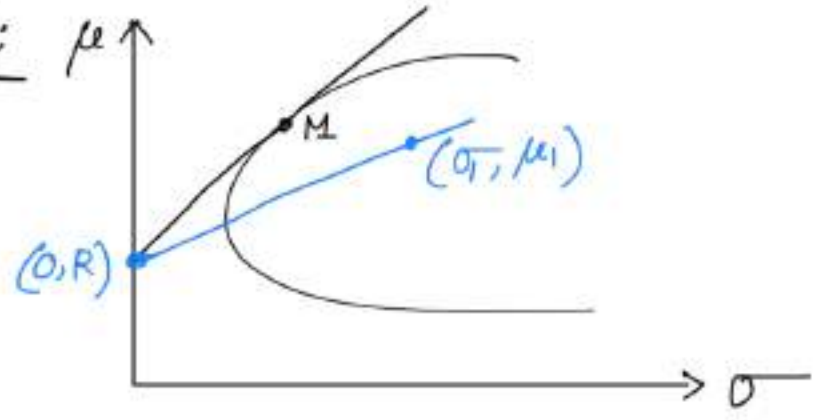
can be constructed as a linear combination of M and R.

<u>Qn:</u> How to construct M:

Theorem 5.2. *Consider a market that consists of one risk-free security with return R, and n risky securities with expected return $\mu = (\mu_1, ..., \mu_n)$ and covariance matrix $\mathbb{C}$. Assume that $R < \mu_{\text{MVP}}$, and if $\mathbb{C}$ is invertible, then the market portfolio M exists and its weights are given by*

$$\boxed{w_M = \frac{(\mu - Ru)\mathbb{C}^{-1}}{(\mu - Ru)\mathbb{C}^{-1}u^\top}.}$$

Proof:



$$\sigma_T = \frac{1}{\sqrt{\omega \mathbb{C} \omega^T}}$$

$$\mu_1 = \mu \omega^T$$

$$\max_{\omega} \text{ slop of the line. } = \frac{\mu\omega^T - R}{\sqrt{\omega \mathbb{C} \omega^T}}$$

$$\text{st} \quad u\omega^T = 1$$

Proof. We look for the V (or equivalently, w) that satisfies:

$$\max_{w} \frac{\mu w^\top - R}{\sqrt{w \mathbb{C} w^\top}} \text{ subject to the constraint } uw^\top = 1.$$

To this end, we define the Lagrangian $\mathcal{L}(w, \lambda) := \frac{\mu w^\top - R}{\sqrt{w\mathbb{C}w^\top}} - \lambda(uw^\top - 1)$.
By calculating the FOC, we derive

$$\begin{aligned} \nabla\mathcal{L} = 0 &\Leftrightarrow \frac{\mu[w\mathbb{C}w^\top]^{\frac{1}{2}} - (\mu w^\top - R)[w\mathbb{C}w^\top]^{-\frac{1}{2}} w\mathbb{C}}{w\mathbb{C}w^\top} - \lambda u = 0 \\ &\Leftrightarrow \mu - \lambda[w\mathbb{C}w^\top]^{\frac{1}{2}} u = \frac{\mu w^\top - R}{w\mathbb{C}w^\top} w\mathbb{C}. \end{aligned} \tag{10}$$

By multiplying with $w^\top$ on the right and using the constraint, we get

$$\lambda = \frac{R}{[w\mathbb{C}w^\top]^{\frac{1}{2}}}.$$

Plug in back to (10), we have

$$\mu - Ru = \gamma w \mathbb{C}, , \tag{11}$$

where $\gamma := \frac{\mu w^\top - R}{w\mathbb{C}w^\top}$ (observe that γ depends on w).
Multiplying by $\mathbb{C}^{-1}u^\top$ on the right, we get

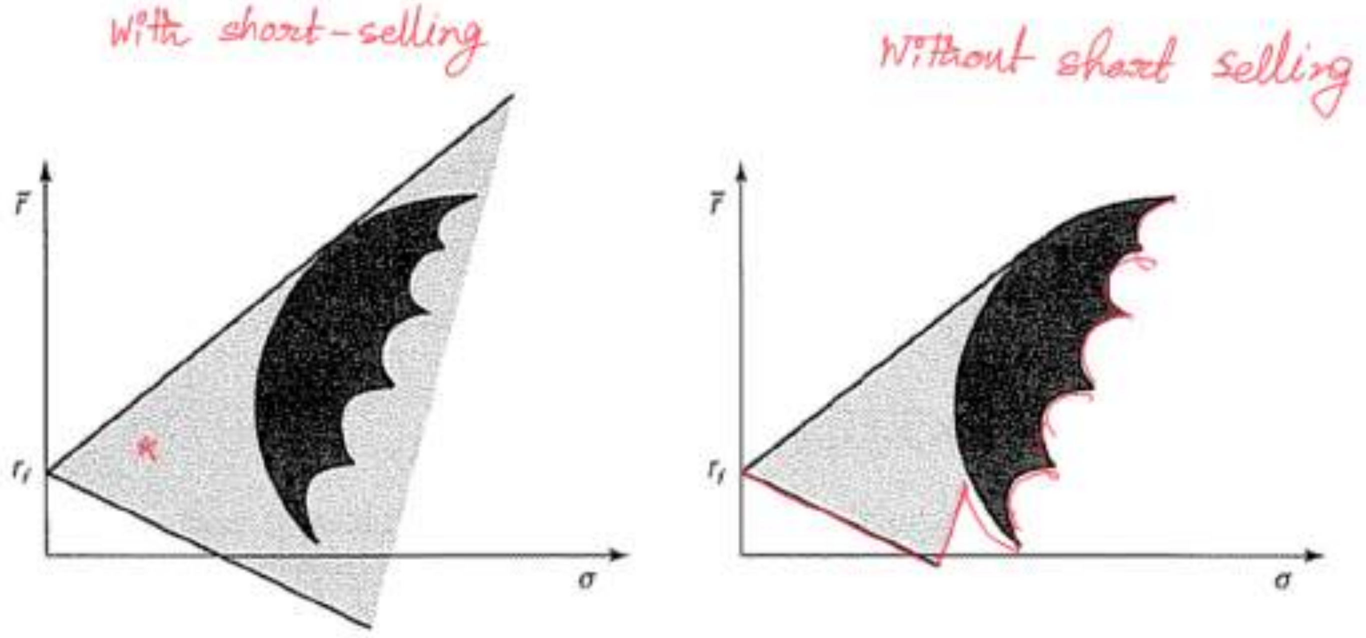
$$\gamma = (\mu - Ru)\mathbb{C}^{-1}u^\top. \tag{12}$$

Note that the RHS is independent of w now.

To solve w from (11), we have to make sure that $\gamma \neq 0$. Recall that

$$\begin{aligned} \mu_{\text{MVP}} = w_{\text{MVP}}\mu^\top = \frac{u\mathbb{C}^{-1}\mu^\top}{u\mathbb{C}^{-1}u^\top} = \frac{\mu\mathbb{C}^{-1}u^\top}{u\mathbb{C}^{-1}u^\top} &\overset{\text{Assumption}}{>} R \\ \Leftrightarrow \mu\mathbb{C}^{-1}u^\top - Ru\mathbb{C}^{-1}u^\top > 0 &\Leftrightarrow \gamma > 0. \end{aligned}$$

Combine (11) and (12) to get the required form. $\square$



(a) (b)

FIGURE 6.13 Effect of a risk-free asset. Inclusion of a risk-free asset adds lines to the feasible region (a) If both borrowing and lending are allowed, a complete infinite triangular region is obtained (b) If only lending is allowed, the region will have a triangular front end, but will curve for larger σ

Does the market port always exists ?

→ if $\mathbb{C}$ is invertable and $R < \mu_{MVP}$ then Yes.

→ And it's unique (given $\mathbb{C}$)

No way of drawing a tangent to the Markowitz bullet.

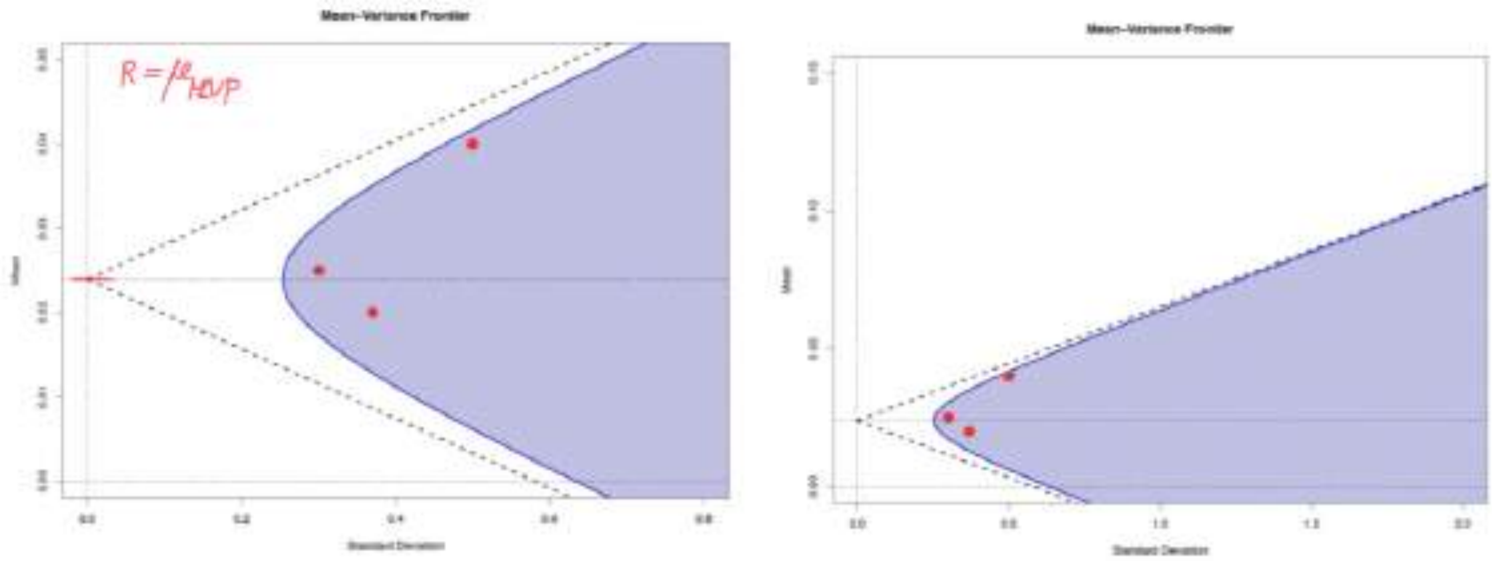


Figure 5: Left/ Right: Non-existence of tangency portfolio.

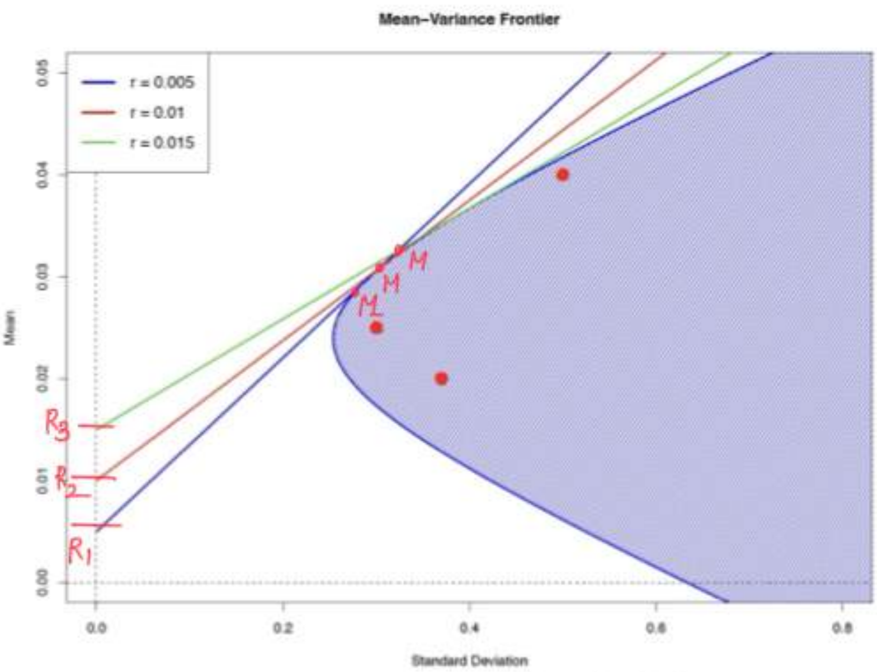


Figure 6: Market portfolio for various risk-free returns R.

Why the name Market portfolio?

→ It contains every risky asset (N many).

→ The weights of asset i is equal to their relative shares in the market.

Capital Market line (CML)

Assumptions:

- No transaction cost
- ~~Every~~ investor has access to same RF R.
- Imp. every investor has the same exp return (μ) and same cov matrix ($\mathbb{C}$).
- $R < \mu_{MVP}$
- all assets are inf. divisable.

What happens under all the assumptions?

→ One fund theorem says: everyone in the market will invest in single risky fund M and RF asset R.

→ The single risky asset is M.

→ Everyone has access to the same M.

Theorem: The CML satisfies.

$$\boxed{\mu_V = R + \underbrace{\frac{\mu_M - R}{\sigma_M}}_{\text{Risk premium.}} \sigma_V}$$

← CML (set of all efficient port).

Risk-premium : $\dfrac{\mu_M - R}{\sigma_M}\,\sigma$

↳ The extra return one gets for taking Risk.

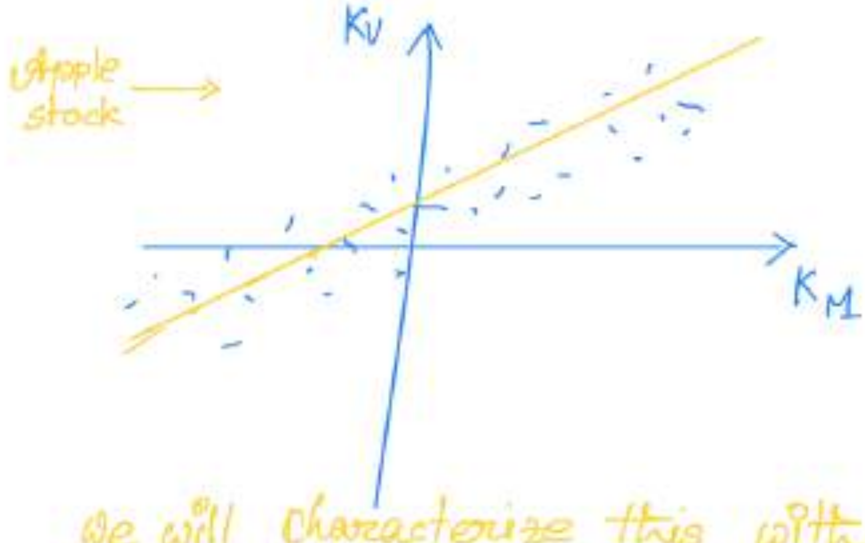
Price of Risk : $\dfrac{\mu_M - R}{\sigma_M}$

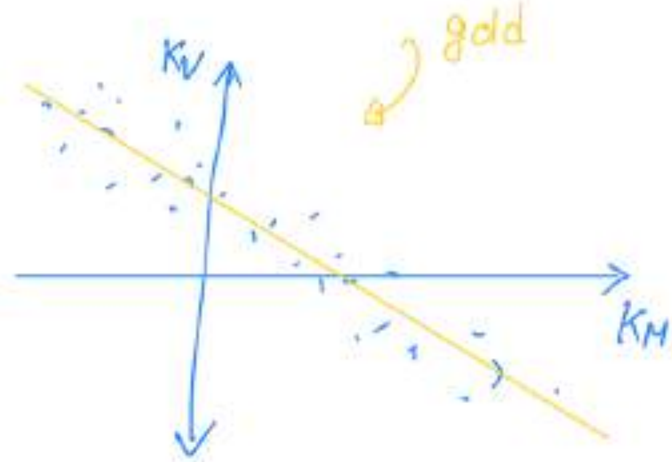
⟶ If we increase risk then the total exp-return also increases.

Capital Asset Pricing Model (CAPM)

- CML only gives the risk-exp return relationship for EP.

It does not show how the μ of an individual asset relates to its σ.

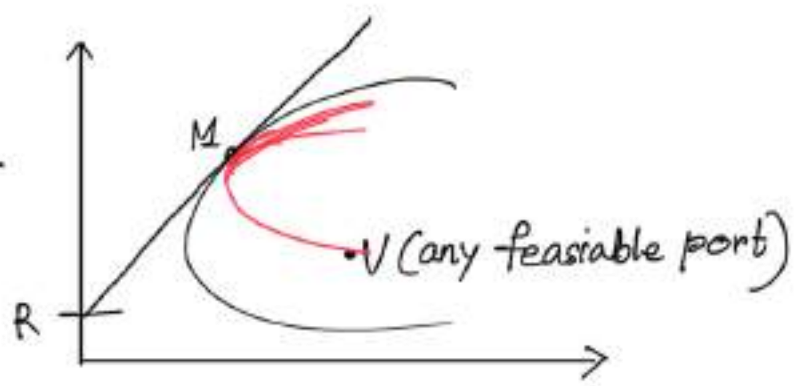




We will characterize this with Beta of a portfolio.

Note: CAPM is applicable to any feasiable portfolio.

Fig*



(Please see the details on lecture note).

Is CML still tangent to the Red curve?

→ Yes.

<u>Def^n</u> (Beta of a portfolio)

Beta factor of port V is given by

$$\boxed{\beta_V = \frac{Cov(K_V, K_M)}{\sigma_M^2}}$$

Using the arguments from fig ⊛ one can construct the eq for CAPM: (See lecture note).

<u>Theorem:</u> Suppose $R < \mu_{MVP}$. <u>Under the assumption that Market portfolio is efficient</u>, then the exp return μ_V for any feasiable portfolio V is given by CAPM:

$$\boxed{\mu_V = R + \beta_V(\mu_M - R)}$$

<u>Remark:</u>

$$\beta_M = 1 \qquad\qquad \beta_V = \frac{Cov(K_V, K_M)}{\sigma_M^2}$$

$$\beta_R = 0$$

Summarizing assumptions of CAPM:

- All investors select portfolios by mean-variance optimization.
- All investors have the same investment horizon.
- All investors use the same mean return vector and covariance matrix.
- Individual investors may have different levels of risk aversion.
- Assets are infinitely divisible.
- No taxes and transaction costs.
- The market is in equilibrium.

Week – 9

Game Theory

Remember to submit "Tutorial Suggession" on KEATs by today.

What is a game?

Def^n

A game in strategic form (Normal form) is a tuple $G = \left(N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$ where

$N :=$ set of all players in the game. (for us 2-players) say $N = \{1, 2\} / \{P1, P2\}$

$S_i :=$ set of all actions/strategies for player i.

$$S := \prod_{i \in N} S_1 \times S_2 \times \cdots \times S_n$$

$\hookrightarrow$ set of all strategies in the game G.

$$u_i : S \longrightarrow \mathbb{R} \quad \forall i \in N$$

$\hookrightarrow$ the utility/payoff for player i.

Example: (Matching Pennies)

two players
- Player 1 (P1)
- Player 2 (P2)

We toss a coin $<\begin{matrix} H \\ T \end{matrix}$

P1 : wants to guess what P2 is going to say (Match)

P2 : wants to guss the opposit of P1 (miss-match)

We use a matrix to represent the game :

P2

P1	h	t
H	1,0	0,1
T	0,1	1,0

$G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$

$N = \{1,2\} / \{P1, P2\} / \{\text{Row}, \text{Column}\}$

$S_1 = \{H, T\}$

$S_2 = \{h, t\}$

The set of all strategy for the game $S = S_1 \times S_2$

$= \{H, T\} \times \{h, t\}$

$= \{H \times h, H \times t, T \times h, T \times t\}$

$u_1 : S \longrightarrow \mathbb{R}$

$H \times h \longmapsto 1$

$H \times t \longmapsto 0$

$T \times h \longmapsto 0$

P2

$u_1(H \times h)$

	h	t
H	1,0	0,1

$T \times t \longmapsto 1$ $\quad$ P1

T	0,1	1,0

$u_2(H \times h)$

$u_2 : S \longrightarrow \mathbb{R}$

$H \times h \longmapsto 0$

$H \times t \longmapsto 1$

$T \times h \longmapsto 1$

$T \times t \longmapsto 0$

Def^n (Finite Game)

We say a game G if N and S both are finite.

players. $\quad$ set of all strategies.

Def^n: (Pure) Nash - Equilibria (NE)

A strategy profile $s^* \in S$ is a Nash Equilibria (NE) if

an element of S (last ex: $H \times t$. Not st. profile $h \times T$ or $h \times h$)

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i$$

$$s^* = \underset{\in S_1}{s_1^*} \times \underset{\in S_2}{s_2^*} \cdots \times \underset{\in S_n}{s_n^*}$$

$$s_{-i}^* = s_1^* \times \cdots s_{i-1}^* \times - \times s_{i+1}^* \cdots \times s_n^*$$

$$(t_i, s_{-i}^*) := s_1^* \times s_2^* \cdots \times s_{i-1}^* \times t_i \times s_{i+1}^* \cdots \times s_n^*.$$

Nothing but saying "no player has any incentive to move (to a different strategy)"

--- profitable for P2

--- Profitable for P1

	h	t
H	1,0	0,1
T	0,1	1,0

This game does not have a (pure) NE

Ex 2: (Prisoners' dilemma)

Cop (with police)

		P2	
		Cop	¬Cop
P1	Cop	−3, −3	0, −10
	¬Cop	−10, 0	−2, −2

(C, C) is a Pure NE ✓

No other pure NE (check)

→ Split or Steal

→ Stage Hunt

→ Battle of sexes.

} — (HW) try to find pure NE

Remark: for a finite game G one can have NO pure NE, or have 1 pure NE, or have >1 pure NE.

Def^n: (Dominated strategy).

A strategy $s_i \in S_i$ for player i in a game G

dominated (also known as strictly dominated) if $\exists\, t_i \in S_i$

st:

$$u_i(s_{-i}, t_i) > u_i(s_{-i}, s_i)$$

Def^n: (Weakly Dominated strategy) ← unless explicitely mentioned we will not use.

A strategy $s_i \in S_i$ for player i in a game G dominated (also known as strictly dominated) if $\exists\, t_i \in S_i$

st:

$$u_i(s_{-i}, t_i) \geq u_i(s_{-i}, s_i)$$

~~Imp~~ Theorem: (Dominance Principal)

Let $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}\right)$.
Let $d_j \in S_j$ be a dominated strategy of player j.

Define:

$$G' = \left(N, \{S_i'\}_{i\in N}, \{u_i'\}_{i\in N}\right)$$

$$S_i' = \begin{cases} S_i & \text{for } i \neq j \\ S_j \setminus \{d_j\} & i = j \end{cases}$$

and u_i' is the restriction of u_i to S_i'.

Then every pure NE of G is in S'.

$\Rightarrow$ if $s \in S'$ is a pure NE in $G' \Longleftrightarrow s$ is a pure

$\Rightarrow$ If $s \in S$ is a pure NE in ... $\Longleftrightarrow$ s is a pure

NE in G.

<u>Theorem</u>: Fix a finite game G. and let G_1 be a game that is

① is the result of repeated elimination of dominated strategies from G.

imp ② <u>NO</u> dominated strategies left.

Say G_2 be another such game.

Then: $\boxed{G_1 = G_2}$

and NE of G_1 = NE of G_2 = NE of G.

<u>Example:</u>

$G^* =$

P1 ↓ \ P2 →	X	Y	Z
A	4,3	5,1	6,2
B	2,1	8,4	3,6
C	3,0	9,6	2,8

Z Y

2 > 1

6 > 4

8 > 6

$Z \succ Y$

dominated st.

⇓

P1 \ P2	X	Z
A	4,3	6,2
B	2,1	3,6
C	3,0	2,8

4 > 3

6 > 2

$$\Downarrow$$

	X	Z
A		
~~B~~		

$$\Downarrow$$

	X	$\cancel{Z}$
A	4,3	$\cancel{6,2}$

$3 > 2$

$$\Downarrow$$

	X
A	4, 3

$$\Downarrow$$

the pure NE is (A, X)

(theorem)

$\Longrightarrow$ (A, X) is pure NE of G^*.

HW: Reverify (A, X) is the only pure NE of G^* just from the def^n of pure NE.

Remark:

→ Every game may not be reduced to 1×1 game.

→ We can have games where we have no dominated strategy to start with.

Ex:

$P_1 \backslash P_2$	X	Y	Z
~~A~~	~~3, −3~~	~~2, −2~~	~~5, −5~~
B	4, −4	3, −3	6, −6
C	1, −1	4, −4	3, −3

$A \propto B$

$Z \propto X$

We can not reduce further.

Def^n (Dominant strategy): (please see Note)

$$u_i(\underline{s}_{-i}, s_i) > u_i(\underline{s}_{-i}, t_i); \quad \forall\, t_i \in S_i$$

$\iff s_i$ is dominant.

Theorem: If player i has a dominant strategy $d_i \in S$ then ~~every~~ for every NE in the game G

say s^* is NE

then $S^* = S_1^* \times S_2^* \times \cdots \times S_n^*$

and $S_i^* \equiv d_i$

Mixed NE / NE

Notation: Given a set X ; we denote ΔX the set of all prob. distribution over X.

	h (q)	t ($1-q$)
H (p)	1,0	0,1
T ($1-p$)	0,1	1,0

$$S = S_1 \times S_2$$
$$= \{H, T\} \times \{h, t\}$$

$$\Delta S = \left\{ (p, 1-p) \times (q, 1-q) \;\middle|\; p, q \in [0,1] \right\}$$

$$\Delta S_1 = \left\{ (p, 1-p) \;\middle|\; p \in [0,1] \right\}$$

Now $(p, 1-p)$ represent
P1 choses H wp p
P1 choses T wp $(1-p)$

$$\Delta S_2 = \left\{ (q, 1-q) \;\middle|\; q \in [0,1] \right\}$$

for ex: take $p = 0,\ q = 0$

$$\Longrightarrow \{ (0,1) \times (0,1) \} \Longrightarrow \text{our old } (T, t)$$

Week — 10

Math - Finance — II

HW 10 : (Week 10 + Week 11)

Mixed NE

Recall: Given a finite set X we donate ΔX to be the distribution over X.

say $S_1 = \{H, T\}$ | $S_2 = \{H, T\}$

$$\longrightarrow \Delta S_1 = \left\{ (\underset{H}{p}, \underset{T}{\underbrace{1-p}}) \;\middle|\; \underset{\text{prob of } H}{p \in [0,1]} \right\}$$

$$\Delta S := \Delta S_1 \times \Delta S_2 \quad \text{for Player 1}$$

$$= \left\{ (p, 1-p) ; (q, 1-q) \;\middle|\; P_1, P_2 \in [0,1] \right\}$$

→ $p :=$ prob of H for player 1.
$q :=$ ——————— 2.

	h	t
H	1,0	0,1
T	0,1	1,0

For example:

$$\sigma = \left\{ \left(\frac{1}{5}, \frac{4}{5}\right) ; \left(\frac{2}{3}, \frac{1}{3}\right) \right\}$$

sum upto 1

Remember: A pure strategy was (T, h)

p1 choses T wp 1

(0,1)

This is $\sigma = \{(0,1);(1,0)\}$

Linking NE of G to a pure NE of $\hat{G}$.

$$G = (N, \{S_i\}, \{u_i\}) \text{ is finite game.}$$

Define $\hat{G} = (\hat{N}, \{\hat{S}_i\}, \{\hat{u}_i\})$

$$\hat{N} = N$$

$$\hat{S}_i = \Delta S_i \quad \text{(Note not finite anymore)}$$

$$\hat{u}_i : \Delta S_1 \times \Delta S_2 \cdots \times \Delta S_N \longrightarrow \mathbb{R}$$

$$\sigma = \sigma_1 \times \sigma_2 \cdots \sigma_N \longmapsto \underbrace{\mathbb{E}_{s \sim \sigma}(u_i(s))}$$

"this is exp payoff when exp is taken wrt the mixed strategy σ.

Then any pure NE of $\hat{G}$ is a NE of the game G.

How to calculate $u_i(\sigma)$ for any $\sigma \in \Delta S_1 \times \Delta S_2 \cdots \Delta S_N$.

Example:

prob $q = 1$ prob $1-q = 0$

prob $\overset{1/3}{\overset{\shortparallel}{p}}$ to chose $\rightarrow$ H

prob $\overset{2/3}{(1-p)}$ $\rightarrow$ T

	h	t
H	1,0	0,1
T	0,1	1,0

Say $\sigma = \left\{\left(\frac{1}{3}, \frac{2}{3}\right); (1,0)\right\}$

payoff for (H,h)   prob of H for P1   Prob of h of P2

$$u_1(\sigma) = 1 \times \frac{1}{3} \times 1 + 0 \times \frac{1}{3} \times 0 + 0 \times \frac{2}{3} \times 1 + 1 \times \frac{2}{3} \times 0$$

player 1

$$= \underbrace{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}}_{\text{P1's strategy}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{P2's strategy.}}$$

the first components of payoff matrix

$$u_2(\sigma) = \left(\tfrac{1}{3} \ \tfrac{2}{3}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Imp

Theorem (Nash 1950)

Every finite game has a NE.

Note:
- A game may not have a pure NE
- A game can have more than 1 NE. Even more than 1 pure NE

Proof: The proof is by Brouwer's fixed point theorem.

Theorem 4.2 (Brouwer's Fixed Point Theorem). *Let X be a compact convex subset of $\mathbb{R}^d$. Let $T : X \to X$ be continuous. Then T has a fixed point. i.e., there exists an $x \in X$ such that $T(x) = x$.*

A simple application: If you are in a room and hold a map of the room horizontally, then there is a point in the map that is exactly above the point it represents.

How to find set of all NE of a Game.

Best Responce

Let σ be any NE in the game G.

We say $\boxed{s_i \in S_i}$ is a best response to σ_{-i} if

$$\forall t_i \in S_i \; : \quad \boxed{u_i(\sigma_{-i}, s_i) \geq u_i(\sigma_{-i}, t_i)}$$

Note: It is not difficult to generalize this to ΔS_i.

Proposition 4.3. *Let $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$ be a finite strategic game. Then, $\sigma^* = (\sigma_1^*, \cdots, \sigma_N^*)$ is a mixed Nash equilibrium of G if and only if for every player i, every pure strategy s_i in the support of σ_i^* (i.e., any s_i to which σ_i^* assigns positive probability) is a best response to σ_{-i}^*.*

Proof. Suppose $s_i \in S_i$ is in the support of σ_i^*, but is not a best response to σ_{-i}^*, and let $t_i \in S_i$ be some best response to σ_{-i}^*. We will prove the claim by showing that t_i is a profitable deviation for i.

Let $C = u_i\left(\sigma_{-i}^*, t_i\right)$. Then $u_i\left(\sigma_{-i}^*, r_i\right) \leq C$ for any $r_i \in S_i$, and $u_i\left(\sigma_{-i}^*, s_i\right) < C$. It follows that

$$\begin{aligned} u_i\left(\sigma^*\right) &= \sum_{r_i \in S_i} \sigma_i^*(r_i)\, u_i\left(\sigma_{-i}^*, r_i\right) \\ &< C \end{aligned}$$

and so t_i is indeed a profitable deviation, since it yields utility C for i.

It follows that if σ^* is an equilibrium then $u_i\left(\sigma_{-i}^*, s_i\right)$ is the same for every s_i in the support of σ_i^*. That is, i is indifferent between all the pure strategies in support of her mixed strategy. $\square$

It simply says a NE is something like <u>NO Profitable deviation</u>

Def^n: (Completely mixed strategy)

We say $\sigma_i \in \Delta S_i$ is a completely mixed strategy

$\quad \longrightarrow$ strategy for Player i

if each $s_i \in S_i$ are assigned strictly +ve prob.

Ex: $\sigma_i = \left(\frac{1}{3} \ \frac{2}{3}\right)$ is a completely mixed st. (CM)

$\sigma_i = \left(0 \ 1\right)$ is a pure strategy.

A strategy does not have to be either pure or CM

$$\sigma_i = \left(\frac{1}{3} \ 0 \ 0 \ 0 \ \frac{1}{3} \ \frac{1}{3}\right)$$

Often a strategy is called "Randomized Strategy" if it is not a pure strategy.

$$\sigma = \sigma_1 \times \sigma_2$$
$$= \left\{\left(\tfrac{1}{3}, \tfrac{2}{3}\right) ; \left(1, 0\right)\right\}$$

- σ as a strategy profile is Not CM

- σ_1 : P1's strategy is
- σ_2 : P2's strategy is pure strategy.

Example: set of all NE

Take

$G =$

	A	B
a	1,1	1,0
b	0,1	4,4

Qn: We want to calculate set of all NE in G.

Step — 1 : check for row/column domination.

For this game NO domination.

$$\left.\begin{array}{l} \text{if } a \gg b \\ \text{or } b \gg a \\ \text{or } A \gg B \\ \text{or } B \gg A \end{array}\right\} \Rightarrow \text{domination.}$$

step — 2 Find set of all pure st (one can skip)

	A	B
a	1,1	1,0
b	0,1	4,4

- take $(a,A) = \{(1,0);(1,0)\}$ → Best response. → No profitable diviation ⇒ pure NE

- $(b,B) \Longrightarrow$ pure NE

- $(a,B) \Longrightarrow$ NOT pure NE

- $(b,A) \Longrightarrow$ NOT pure NE.

Step—3 Find all NE using best response.

from Nash's theorem we know $\exists\ \sigma^*$ which is a NE of the game.

Assume σ^* is a NE of the game G.

$$\sigma^* = \left\{ (p^* \quad 1-p^*) \; ; \; (q^* \quad 1-q^*) \right\}$$

$$\begin{array}{ccc} & q^* & 1-q^* \\ & A & B \\ p^* \to a & 1,1 & 1,0 \\ 1-p^* \to b & 0,1 & 4,4 \end{array}$$

Now: Fix player 1.

then $\sigma_1^* = (p^* \quad 1-p^*)$ is a best responce to σ_{-1}^*

$$\Rightarrow u_1(\sigma^*) \geq u_1(\sigma_{-1}^*, x) \quad \forall\, x \in \Delta S_1$$

$$\iff \begin{pmatrix} p^* & 1-p^* \end{pmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} q^* \\ 1-q^* \end{bmatrix} \geq \begin{pmatrix} p & 1-p \end{pmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} q^* \\ 1-q^* \end{bmatrix}$$

$$\forall p \in [0,1]$$

⋮

$$\iff \underline{(p^*-p)\left(1-4(1-q^*)\right) \geq 0} \quad \forall p \in [0,1].$$

$$\boxed{\iff \begin{cases} \underline{\text{Either}} \quad 1-4(1-q^*) = 0 \Rightarrow q^* = 3/4 \; ; \; p^* \in [0,1] \\ \text{or} \quad p^* = 1 \text{ and } 1-4(1-q^*) > 0 \Rightarrow q^* \geq 3/4 \\ \text{or} \quad p^* = 0 \text{ and } 1-4(1-q^*) < 0 \Rightarrow q^* \leq 3/4. \end{cases}}$$

Let's repete for player 2

$$u_2(\sigma^*) \geq u_2(\sigma^*_{-2}, y) \quad \forall y \in \Delta S_2.$$

⋮

$$\boxed{\iff \begin{cases} \text{Either} \quad 1-4(1-p^*) = 0 \Rightarrow p^* = 3/4 \text{ and } q^* \in [0,1] \\ \text{or} \quad q^* = 1 \text{ and } p^* \geq 3/4 \\ \text{or} \quad q^* = 0 \text{ and } p^* \leq 3/4 \end{cases}}$$

say $p^* = 0 \implies q^* = 0.$

$p^* = 1 \implies q^* = 1$

$q^* = 3/4 \implies p^* = 3/4$

and no othere feasible options.

So set of all NE :

① $\{(0 \; 1);(0, 1)\}$

② $\{(1 \; 0);(1 \; 0)\}$

③ $\{(\frac{3}{4} \; \frac{1}{4});(\frac{3}{4} \; \frac{1}{4})\}$

→ NE (mixed/general NE)

→ Pure NE

→ Randomized NE

→ Completely mixed NE

Anothere consequence:

(support prop):

If σ^* is a NE; and $s_i \in S_i$ is assigned >0 prob on σ^*

similarly say $s_i' \in S_i$ is also asigned >0 prob on σ^*

Then: $u_i(\sigma_{-i}^*, s_i) = u_i(\sigma_{-i}^*, s_i')$

Proposition 4.4 (Support Property). *In any mixed equilibrium σ^*, all pure strategies s_i (for both players) that receive positive probability under σ^* yield equal expected payoffs:*

$$u_i(\sigma_{-i}, s_i) = u_i(\sigma_{-i}, s_i') \quad \textit{for all } s_i, s_i' \in Supp(\sigma_i^*).$$

Otherwise, the player could shift probability towards the more profitable pure strategy.

This leads to the so-called Indifference Principle, often used to compute mixed equilibria in 2×2 games.

Note: (Added after class).

there might have been some confusion with notation during class. The following three terms are all the same.

$$u_i(\sigma^*) = u_i((\sigma_1^* \cdots\cdots \sigma_n^*)) = u_i(\sigma_{-i}^*, \sigma_i)$$

Week — 11

Game Theory

Zero-Sum Games

A two player game $G = (\{P1, P2\}, \{S_1, S_2\}, \{u_1, u_2\})$ finite.

is called zero-sum if $u_1 + u_2 = 0$

$$\begin{cases} u_1 : S_1 \times S_2 \longrightarrow \mathbb{R} \\ u_2 : S_1 \times S_2 \longrightarrow \mathbb{R} \end{cases}$$

$$u_1 + u_2 = 0 \iff \forall\ s_1 \times s_2 \in S_1 \times S_2$$
$$u_1(s_1, s_2) = -u_2(s_1, s_2)$$

Example:

$G =$

1, −1	−1, 1
2, −2	4, −4
3, −3	−2, 2

$\xrightarrow[\text{game}]{\text{a zero-sum}}$

1	−1
2	4
3	−3

correction should be −2

standard notation.

Remark: This is a game with co-operative component.

4, 3	0, 0
0, 0	3, 4

A zero-sum game can never have a co-operative component.

$\Longleftrightarrow$ P1' Gain $=$ P2's loss.

and P1' loss $=$ P2's gain.

(*) Player 1's prospective:

P1 knows P2 will try to max P2's profit.

$\Longrightarrow$ P1 will try to prepare for worst possiable outcome.

"Worst possiable outcome" $=$ given P1's strategy P2 will chose the strategy $\sigma_2 \in \Delta S_2$ which gives max $u_2 \Leftrightarrow$ min u_1.

$$u_g(\sigma_1) = \min_{\sigma_2 \in \Delta S_2} u_1(\{\sigma_1, \sigma_2\})$$

The guaranteed payoff P1 will recieve by playing the strategy $\sigma_1 \in \Delta S_1$

Now P1 will try to maximize their own payoff.

$$\sigma_1^* = \underset{\sigma_1 \in \Delta S_1}{\operatorname{argmax}} \; u_g(\sigma_1)$$

$$= \underset{\sigma_1 \in \Delta S_1}{\operatorname{argmax}} \; \min_{\sigma_2 \in \Delta S_2} \; u_1(\{\sigma_1, \sigma_2\}).$$

(*) Same argument can be done from a P2's prospective.

$$\sigma_2^* = \arg\min_{\sigma_2 \in \Delta S_2} \max_{\sigma_1 \in \Delta S_1} u_1(\{\sigma_1, \sigma_2\})$$

should be $\underline{v}$

$$\overline{v} = \max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u_1(\{\sigma_1, \sigma_2\}) \geqslant \min_{\sigma_2 \in \Delta S_2} \max_{\sigma_1 \in \Delta S_1} u_1(\{\sigma_1, \sigma_2\}) =: \underline{v}$$

should be $\leqslant$ should be $\overline{v}$

The one in typed note was correct.

<u>Theorem: (Von-Neumann)</u> In a zero-sum game

$$\overline{v} = \underline{v}.$$

$\overline{v} = \underline{v}$ is also known as value of the game v.

<u>Notations for zero-sum game</u>

- A zero-sum game is always represented by matrix.

$$A = (a_{ij})_{i=1, j=1}^{m,n}$$

- P1 — always represents row player.
- P2 — always the column player.
- $S_1 = \{1, \dots, m\}$
- $S_2 = \{1, \dots n\}$
- $S = S_1 \times S_2 = \{(i \times j) \mid i \in \{1, \dots m\}\}$

$S = S_1 \times S_2 = \{(i \times j) \mid \quad j \in \{1, \ldots n\}\}$

- a_{ij} is the payoff P1 gets when P1 plays $i \in S_1$
  P2 plays $j \in S_2$

- $(-a_{ij})$ ...... P2 ....

- u = utility of player 1 $(= u_1)$

  we write $-u$ to represent u_2.

- The strategy of P1 will be written as $x = (x_1 \ldots x_m)$
  with $x_1 + x_2 \ldots + x_m = 1$

- The strategy of P2 will be written as $y = (y_1 \ldots y_n)$
  with $y_1 + \quad \ldots + y_n = 1$

- $$u(x, y) \overset{\text{(from last class)}}{=} \underset{1 \times m}{x} \; \underset{m \times n}{A} \; \underset{1 \times n}{y^T} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$$

Imp

Theorem: Take G to be finite zero sum game with payoff matrix $A_{m \times n}$.

Then:

① Every NE of the game G : denot it as $\sigma^* = (x^*, y^*)$

① Every NE ...

is a min-max strategy.

② And they matches with value of the game.

Mathematicall;

every NE $\sigma^* = (x^*, y^*)$; we have.

$$
\begin{array}{ccc}
\max\limits_{x} \min\limits_{y} u(x,y) & = & u(x^*, y^*) \\
\| & & \| \\
\min\limits_{y} \max\limits_{x} u(x,y) & & v
\end{array}
$$

Note:

- A game can have more than 1 NE
- but the payoff from each NE are the same.
- payoff from any NE matches with value of the game (v)
- The value of the game is <u>unique</u>

Remark! $v(G_c) = \max \min = \min \max$.

Theorem: if $\sigma^* = (x^*, y^*)$ is a NE then:

Minimum entry of $\underbrace{x^* A}_{(1\times m)\times(m\times n) = 1\times n}$ = max entry of $\underbrace{A y^{*T}}_{(m\times n)(1\times n)^T = m\times 1}$.

Proposition: if $\sigma^* = (x^*, y^*)$ is a NE of the game $A = (a_{ij})$

then: $x^* A \geqslant v \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{1 \times n}$.

$A(y^*)^T \leqslant v \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$.

Findin 1 NE in a game A is a linear-problem and can be compluted in polynomial time by solving a convex optization problem.

$$
\begin{aligned}
\text{(Primal)} \quad & \max_{x \in \mathbb{R}^m,\, v \in \mathbb{R}} \quad \underline{v} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_i A_{ij} \geq v, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} x_i = 1, \quad \longleftarrow \text{valid strategy} \\
& x_i \geq 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

we assign $\geqslant 0$ prob. $\longrightarrow$ $x_i \geq 0$

The dual formulation (again a convex optimization problem) of the above problem is as follows:

$$
\begin{aligned}
\text{(Dual)} \quad & \min_{y \in \mathbb{R}^n,\, w \in \mathbb{R}} \quad w \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_j A_{ij} \leq w, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} y_j = 1, \\
& y_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

"Symmetric game is a fair game"

Def^n: A fair game is a game with value of the game $v = 0$.

Def^n: A symmetric game is a game with anti-symmetric payoff matrix. i.e

$$\boxed{A = -A^T}$$

Theorem: A symmetric game is alway a fair game. i.e. $\boxed{v = 0}$.

Proof: A simple consequence of the lemma below.

Lemma: For any payoff matrix A one has

$$v(A) = -v(-A^T)$$

Proof:
$$v(-A^T) = \max_{x \in S_1} \min_{y \in S_2} x(-A^T)y$$

$$= \max_{x} \min_{y} -xA^Ty$$

$$= \max_{x} -\left(\max_{y} \quad xA^Ty\right)$$

$$= -\min_{x} \max_{y} \underbrace{(xA^Ty)^T}_{yAx^T}$$

$$= - \min_{\alpha} \max_{\beta} \beta A \alpha^T$$

$$= - \min_{y} \max_{x} x A y^T$$

$$= - v(A)$$

Saddel point :

- This is another way to find value of the game v

→ Look at the min entry of the column maxima.

if they coincide then this gives value of the game.

$$A = \begin{array}{cccc|l} & & & & \text{Row minima} \\ \left(\begin{matrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{matrix}\right)_{2\times 3} & & & & \begin{matrix} 6 \\ 2 \end{matrix} \\ \text{column maxima} \quad \underline{6} \quad 12 \quad \underline{6} & & & & \end{array}$$

$\begin{array}{cc} & 6 \\ 6 & \end{array}$ = value of the game

Note: we can have $0, 1, >1$ saddle points.

Theorem: every saddel point has same matrix entry = value of the game.

The saddle point corresponds to pure NE.

Ex: Above game.

$$\{(1\ \ 0); (\underline{1}\ \ 0\ \ 0)\} \text{ is a pure NE}$$

$$\{(1\ \ 0); (0\ \ 0\ \ \underline{1})\} \text{ is a pure NE.}$$

We may not have saddle point: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Rock - Paper - Scissor.

$R \prec P$

$R \succ S$

R P S

$P \prec$ ...

$$A = \begin{array}{c} R \\ P \\ S \end{array} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$A = -A^T \Rightarrow$ symmetric game.

$\Rightarrow v = 0.$

Say $x^* = (x_1^* \ \ x_2^* \ \ x_3^*)$ is optimal for P1.

$$\Rightarrow \quad x^* A \geqslant \underset{\overset{\shortparallel}{0}}{v} [1 \quad 1 \quad 1]$$

$$\Rightarrow \begin{pmatrix} x_1^* & x_2^* & x_3^* \end{pmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \geqslant [0 \quad 0 \quad 0]$$

$$\Rightarrow \begin{cases} x_2^* - x_3^* \geqslant 0 \\ -x_1^* + x_3^* \geqslant 0 \\ x_1^* - x_2^* \geqslant 0 \end{cases} + \left(x_1^* + x_2^* + x_3^* = 1 \right)$$

We simply check for equality:

$$x_1^* = x_2^* = x_3^* = \frac{1}{3}.$$

P2

NE of the game as:

We get $\underline{\underline{a}}$ NE of the game

$$\left\{ \left(\tfrac{1}{3} \;\; \tfrac{1}{3} \;\; \tfrac{1}{3}\right) ; \left(\tfrac{1}{3} \;\; \tfrac{1}{3} \;\; \tfrac{1}{3}\right) \right\}$$

Final exam
- Utility
- Portfolio management without RF
- ——— with RF
- Game theory