Generalized p-th variation and wavelets

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Abstract

We study the generalized p-th variation of a continuous path along a given sequence of partitions and its interplay with wavelet coefficients. Existing characterisations of generalized p-th variation are expressed in terms of a Schauder basis built from the partition sequence itself. We replace these partition-dependent coordinates with path coefficients in a fixed wavelet basis, which provides a common multiscale description of the path and allows comparison across different partition sequences.

We give sufficient conditions, formulated as weighted ℓ^p-norms of the wavelet coefficients, for a path to have bounded generalized p-th variation along a balanced partition sequence, and we show that these conditions depend on the partition only through its mesh. Conversely, along balanced complete refining sequences, the generalized p-th variation controls the same ℓ^p-norms of the wavelet coefficients. Together with classical embeddings, these estimates place the generalized p-th variation space between two Besov spaces at the critical smoothness, with the space of paths of finite classical p-variation lying in between. We conclude the paper by showing that for paths with positive Hölder regularity, the critical exponents measuring roughness through Besov regularity, through classical variation, and through variation along a partition sequence all coincide; in particular, this exponent does not depend on the choice of partition sequence and a higher-order pathwise Itô formula independent of the choice of partition sequence.

Keywords— generalized p-th variation; variation index; wavelet coefficients; Besov spaces; refining sequences; fractional Brownian motion.
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1 Introduction

The quadratic variation along a sequence of partitions was originally introduced by Föllmer in 1981 [10] to prove a pathwise Itô formula. Cont-Perkowski [6] introduced the notion of p-th variation along a partition π as a natural generalization of quadratic variation. As expected the p-th variation appears in the rough pathwise Itô formula as well as a pathwise Itô isometry-type formula for paths/process that are rougher than semimartingales. Explicit examples of paths with a given p-th variation along dyadic partition sequence were constructed in [15], where the Faber-Schauder basis was used for the construction of such paths.

Even though the p-th variation is a natural object arising in the rough Itô formula, as shown by Schied [17], the class of paths with finite quadratic variation along a given sequence of partitions is not even a vector space. In a recent work [7], a class of continuous paths with generalized p-th variation along a prescribed partition sequence π was introduced, denoted by $\mathcal{X}_\pi^p$. This generalized class $\mathcal{X}_\pi^p$ not only has a vector space structure, but is a Banach space when equipped with a suitable norm. In the same paper, under appropriate assumptions on the partition sequence π, the elements of $\mathcal{X}_\pi^p$ are characterized in terms of the coefficients of the Schauder basis associated with π. This partition-dependent Schauder basis was introduced in [4], to provide sufficient conditions under which quadratic variation is invariant across suitably regular partition sequences. While this description is particularly well adapted to a fixed partition sequence, the corresponding Schauder basis, and hence the associated coefficients, depend strongly on the choice of π.

This dependence on the partition sequence makes it difficult to compare generalized p-th variation of a given path along different partition sequences.

The purpose of this work is to replace these partition-dependent coordinates by coefficients with respect to a fixed wavelet basis. Since the wavelet system is chosen independently of the partition sequence, this provides a common multiscale description of the path which can be used to study and compare its p-th variation along different sequences of partitions. Such a partition-independent description is particularly relevant because the dependence of quadratic variation (and hence p-th variation) on the choice of the partition sequence can be extreme. Freedman [11, pp. 47–48] (originally proved by Besicovitch and Schoenberg [1, Theorem 3]) showed in his book that for any continuous function x one can find a sequence of partitions along which the quadratic variation vanishes. More generally, Davis, Obłój, and Siorpaes [8, Theorem 7.1] showed that, for a continuous path whose quadratic variation can be made arbitrarily large on every subinterval, any prescribed increasing function A with $A(0) = 0$ can be obtained as the limit of quadratic variations along a suitable refining sequence of partitions. Their Theorem 7.4 establishes the corresponding result for Brownian motion with random partitions. Related results for generalized p-th variation appear in [7]. This strong dependence on the partition sequence motivates the search for a common system of coordinates that does not itself depend on the partition. Wavelet coefficients provide precisely such a framework.

Besov spaces provide a natural functional framework for describing the global regularity of continuous paths; see, for instance, [14, 18]. One of their main advantages in the present context is that this regularity can be characterized through multiscale coefficients. In particular, classical results of Ciesielski, Kerkyacharian and Roynette [3] characterize Besov spaces in terms of the coefficients of the Faber–Schauder expansion, or equivalently through second-order dyadic

increments. Besov spaces also admit well-known characterizations in terms of wavelet coefficients [14], making them particularly well suited to the partition-independent multiscale approach developed here.

A further connection between Besov regularity and dyadic p- th variation was established by Rosenbaum [16], who obtained a characterization of Besov spaces using dyadic p-th variations. More precisely, for $1/p < s < 1$, the Besov space $B^s_{p,q}([0,1])$ can be characterized by suitable weighted norms of these discrete variations. Our approach builds on this connection by introducing wavelet criteria for generalized p-th variation along more general partition sequences, using coefficients that are independent of the partition. In particular, when the associated sequence of scales has uniformly bounded gaps, the boundedness of our wavelet criterion is equivalent to membership in a classical Besov space.

Our first main result gives sufficient conditions for bounded generalized p-th variation along balanced partition sequences in terms of weighted ℓ^p-norms of wavelet coefficients. These criteria involve the mesh of the partition but not the positions of its points. A fractional Brownian motion example shows that the sufficient condition cannot, in general, be reversed for every choice of the truncation parameter below the critical endpoint. At the critical scale, however, we establish a converse estimate for balanced and complete refining partition sequences. Together with classical embeddings, this gives

$$B^{1/p}_{p,1}([0,1]) \subset \mathcal{V}_{p\text{-var}} \subset \mathcal{X}^p_\pi \subset B^{1/p}_{p,\infty}([0,1]).$$

The Besov bounds are expressed through coefficients in a fixed wavelet basis, independently of the partition sequence.

Our second main result concerns the critical exponents associated with these different notions of regularity. For paths with positive Hölder regularity and balanced refining partition sequences with vanishing mesh and bounded gap, we obtain

$$p^\pi(x) = p_{\text{var}}(x) = p_{\text{Besov},q}(x), \qquad 1 \le q \le \infty.$$

Thus, although generalized p-th variation may depend on the choice of partition sequence, its critical index does not within this class. The equality between the two variation indices was already established in [13]; our contribution is to identify their common value with the critical Besov indices through wavelet coefficients that are independent of the partition sequence.

Preview: The paper is organized as follows. Section 2 recalls the partition sequences, variation spaces, and wavelet framework. Section 3 establishes the wavelet criteria, their Besov interpretation, and the fractional Brownian motion counterexample. Section 4 proves the converse estimate at the critical scale. Section 5 studies the corresponding variation and Besov indices. Section 6 provides a pathwise change of variable formula at every order above the critical one, valid and invariant along every sequence of partitions. The longer proofs are collected in Section 7.

2 Definition and preliminaries

We collect the notation and preliminary results used throughout the paper.

2.1 Sequences of partitions

For a fixed $T > 0$, we shall consider a (deterministic) sequence of partitions $\pi = (\pi^n)_{n\ge 0}$ of $[0,T]$

$$\pi^n = \left(0 = t^n_0 < t^n_1 < t^n_2 < \cdots < t^n_{N(\pi^n)} = T\right),$$

where we denote $N(\pi^n)$ the number of intervals in the partition π^n. By convention, $\pi^0 = \{0,T\}$. For example, the dyadic partition sequence, denoted by $\mathbb{T} \equiv \pi$, contains partition points $t^n_k = kT/2^n$ for $n \in \mathbb{N}$, $k = 0, \cdots, 2^n$.

Definition 2.1 (Refining sequence of partitions). *A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be refining (or nested), if $t \in \pi^m$ implies $t \in \cap_{n\geq m}\pi^n$ for every $m \in \mathbb{N}$. In particular, we have $\pi^1 \subseteq \pi^2 \subseteq \cdots$.*

For a partition sequence $\pi = (\pi^n)_{n\geq 0}$, we write

$$\underline{\pi^n} := \inf_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \qquad |\pi^n| := \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \tag{1}$$

the size of the smallest and the largest interval of π^n, respectively. In the following, we denote $\Pi([0,T])$ the collection of all refining partition sequences π of $[0,T]$ with vanishing mesh, i.e., $|\pi^n| \to 0$ as $n \to \infty$.

Definition 2.2 (Finitely refining sequence of partitions). *A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ in $\Pi([0,T])$ is said to be finitely refining, if there exists a positive integer M such that the number of partition points of π^{n+1} within any two consecutive partition points of π^n is always bounded above by M, irrespective of $n \geq 0$.*

Definition 2.3 (Balanced sequence of partitions). *A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be balanced, if there exists a constant $c > 1$ such that*

$$\frac{|\pi^n|}{\underline{\pi^n}} \leq c \tag{2}$$

holds for every $n \in \mathbb{N}$.

While the finitely refining property controls the number of subintervals created at each refinement step, the balanced property ensures that all intervals of a given partition have comparable lengths.

Definition 2.4 (Sequence of partitions with uniformly bounded gap). *A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to have bounded gap, if there exists a constant $K < \infty$ such that*

$$\frac{|\pi^n|}{|\pi^{n+1}|} \leq K \tag{3}$$

holds for every $n \in \mathbb{N}$.

Definition 2.5 (Complete refining sequence of partitions). *A finitely refining sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be complete refining, if there exist positive constants a and b such that*

$$1 + a \leq \frac{|\pi^n|}{|\pi^{n+1}|} \leq b, \tag{4}$$

holds for every $n \in \mathbb{N}$.

These assumptions will play different roles in the sequel. The balanced property is sufficient for our upper estimates of generalized p-th variation, while the converse estimates require complete refinement. The bounded gap condition will also allow us to compare the wavelet criteria associated with different partition sequences.

2.2 p-th variation

Let $C^0([0,T])$ denote the space of real-valued continuous functions on $[0,T]$.

Definition 2.6. *Fix a partition sequence $\pi = (\pi^n)_{n\geq 0} \in \Pi([0,T])$, $x \in C^0([0,T])$ and $p \geq 1$. For each $n \geq 0$, we denote*

$$[x]^{(p)}_{\pi^n}(t) := \sum_{\ell=0}^{N(\pi^n)-1} \mathbf{1}_{\{t^n_{\ell+1}\leq t\}} \big|x(t^n_{\ell+1}) - x(t^n_\ell)\big|^p, \qquad \forall\, t \in [0,T], \tag{5}$$

the p-th variation of x along the partition π^n. If the limit of $[x]^{(p)}_{\pi^n}(t)$ as $n \to \infty$ exists and is continuous, then we say that x admits finite p-th variation along the partition sequence π, and write its limit as

$$[x]^{(p)}_{\pi}(t) := \lim_{n\to\infty} [x]^{(p)}_{\pi^n}(t), \qquad \forall\, t \in [0,T]. \tag{6}$$

In such case, the convergence is uniform in t and the limit $t \mapsto [x]^{(p)}_{\pi}(t)$ is non-decreasing [6, Definition 1.1 and Lemma 1.3]. We denote by V^p_π the space of such functions x admitting finite p-th variation along π.

Note that functions in V^p_π do not necessarily have finite p-variation in the usual sense. Recall that the p-variation of a function $x \in C([0,T],\mathbb{R})$ is defined as

$$\|x\|_{p-var} := \left(\sup_{P\in\mathcal{P}([0,T])} \sum_{[t_j,t_{j+1}]\in P} \big|x(t_{j+1}) - x(t_j)\big|^p \right)^{\frac{1}{p}},$$

where $\mathcal{P}([0,T])$ denotes the set of all finite partitions of $[0,T]$, see e.g. [9]. The class of all functions with finite p-variation is denoted as $\mathcal{V}_{p-var}$.

A classical example illustrating the distinction between these two notions is Brownian motion B. For any fixed deterministic refining partition sequence π with vanishing mesh, one has $[B]^{(2)}_\pi(t) = t$ almost surely for all $t \in [0,T]$. Nevertheless, Brownian sample paths have infinite classical 2-variation almost surely, that is, $\mathbb{P}(\|B\|_{2-var} = \infty) = 1$.

As shown by [17], the class V^p_π is not a vector space even for $p = 2$. To overcome this limitation, in [7], a generalised p-th variation space was introduced, which turned out to be a Banach space.

Definition 2.7. *For $x \in C^0([0,T])$, $p \geq 1$, and $\pi \in \Pi([0,T])$, we denote*

$$\|x\|^{(p)}_\pi := |x(0)| + \sup_{n\in\mathbb{N}} \Big([x]^{(p)}_{\pi^n}(T)\Big)^{\frac{1}{p}}$$

and consider the subspace of $C^0([0,T])$:

$$\mathcal{X}^p_\pi := \{x \in C^0([0,T]) \,:\, \|x\|^{(p)}_\pi < \infty\}. \tag{7}$$

We say $\mathcal{X}^p_\pi$ is the class of continuous functions with finite (generalized) p-th variation along π.

As discussed in [7], one has

$$\|x\|^{(p)}_\pi < \infty \iff \limsup_n [x]^{(p)}_{\pi^n} < \infty.$$

2.3 Wavelet basis

For simplicity, we restrict ourselves to the interval $[0,1]$ throughout this section, without loss of generality. For $x \in C^0([0,1])$, we first introduce a modification that allows us to extend the function by zero to $\mathbb{R}$ while preserving its continuity and (global) Hölder regularity. We write

$$x(t) = \tilde{x}(t) + y(t),$$

where

$$\tilde{x}(t) = x(0) + (x(1) - x(0))t \quad \text{and} \quad y(t) = x(t) - \tilde{x}(t).$$

Clearly, $y(0) = y(1) = 0$ and $y \in C^\alpha([0,1])$ if and only if $x \in C^\alpha([0,1])$, for $0 < \alpha < 1$.

The affine part does not contribute to the asymptotic p-th variation. Indeed, for every partition $\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = 1\}$, we have

$$\begin{aligned}[\tilde{x}]_{\pi^n}^{(p)}(1) &= \sum_{\ell=0}^{N(\pi^n)-1} \left|\tilde{x}(t_{\ell+1}^n) - \tilde{x}(t_\ell^n)\right|^p \\ &= |x(1) - x(0)|^p \sum_{\ell=0}^{N(\pi^n)-1} |t_{\ell+1}^n - t_\ell^n|^p \le |x(1) - x(0)|^p |\pi^n|^{p-1} \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

as a consequence of $p > 1$ and $|\pi^n| \to 0$. Moreover, by the triangle inequality,

$$\left|\left([x]_{\pi^n}^{(p)}(t)\right)^{1/p} - \left([y]_{\pi^n}^{(p)}(t)\right)^{1/p}\right| \le \left([\tilde{x}]_{\pi^n}^{(p)}(1)\right)^{1/p} \longrightarrow 0,$$

uniformly for $t \in [0,1]$. Consequently, the boundedness of the discrete (generalized) p-th variations along $\pi = (\pi^n)_n$ is unchanged if x is replaced by y. In particular,

$$x \in \mathcal{X}_\pi^p \quad \Longleftrightarrow \quad y \in \mathcal{X}_\pi^p.$$

The same holds for V_π^p, and their limiting p-th variations, whenever they exist, coincide.

We may therefore restrict ourselves to functions vanishing at the endpoints when studying asymptotic p-th variation. We extend y by zero outside $[0,1]$. Since $y(0) = y(1) = 0$, this extension, still denoted by y, belongs to $C_c^0(\mathbb{R})$. Moreover, if $x \in C^\alpha([0,1])$ for some $0 < \alpha < 1$, then $y \in C_c^\alpha(\mathbb{R})$.

We consider a compactly supported orthonormal wavelet basis associated with a multiresolution analysis. Let ψ and φ denote the mother wavelet and the corresponding scaling function, respectively. We assume that both belong to $C^1(\mathbb{R})$ and that ψ has at least one vanishing moment. Up to an integer translation, we assume that

$$\operatorname{supp} \psi \subset [0, R]$$

for some $R > 0$. We use the L^∞-normalization

$$\psi_{j,k}(t) = \psi(2^j t - k)$$

and define the corresponding wavelet coefficients by

$$c_{j,k} := 2^j \int_{\mathbb{R}} y(t)\psi(2^j t - k)\, dt = 2^j \int_0^1 \big(x(t) - \tilde{x}(t)\big)\psi(2^j t - k)\, dt.$$

Since $\int_{\mathbb{R}} \psi(t)\, dt = 0$, we obtain

$$|c_{j,k}| \le \int_{\mathbb{R}} \left|y(2^{-j}(u+k)) - y(2^{-j}k)\right| |\psi(u)|\, du \le \|\psi\|_{L^1} \omega_y(R2^{-j}), \tag{8}$$

where ω_y denotes the modulus of continuity of y. In particular, if $x \in C^\alpha([0,1])$ for some $0 < \alpha < 1$, then

$$|c_{j,k}| \lesssim 2^{-\alpha j},$$

where the implicit constant depends on x, ψ and α.

With the normalization introduced above, the wavelet decomposition of y is given by

$$y(t) = g(t) + \sum_{j \ge 0} \sum_{k \in \mathbb{Z}} c_{j,k} \psi(2^j t - k),$$

where

$$g(t) = \sum_{k\in\mathbb{Z}} C_k \varphi(t-k), \qquad C_k = \int_{\mathbb{R}} y(t)\varphi(t-k)\,dt,$$

is the low-frequency part. Since y and φ are compactly supported, only finitely many coefficients C_k are nonzero. In particular, $g \in C^1(\mathbb{R})$.

Consequently, for every $t \in [0,1]$,

$$x(t) = \tilde{x}(t) + g(t) + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j t - k). \tag{9}$$

The low-frequency part does not contribute to the asymptotic p-th variation for $p > 1$. Indeed,

$$[g]^{(p)}_{\pi^n}(1) \leq \|g'\|^p_\infty \sum_{\ell=0}^{N(\pi^n)-1} |t^n_{\ell+1} - t^n_\ell|^p \leq \|g'\|^p_\infty |\pi^n|^{p-1} \xrightarrow[n\to\infty]{} 0.$$

Hence, by the triangle inequality, the asymptotic p-th variation of y is unchanged if we subtract g. Together with the preceding argument for the affine part, this shows that we may restrict ourselves to the wavelet component

$$\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j t - k)$$

when studying generalized p-th variation.

Finally, we introduce the following (partition independent) object, which will be used throughout the paper:

$$A^{(p)}_j := \Big(\sum_{k\in\mathbb{Z}} |c_{j,k}|^p\Big)^{1/p}, \qquad j = 0, 1, \ldots. \tag{10}$$

Since y and ψ are compactly supported, only finitely many coefficients $c_{j,k}$ are nonzero at each scale. In particular, $A^{(p)}_j$ is well-defined. Note also that $A^{(p)}_j$ depends only on x and the chosen wavelet basis, and not on the partition sequence π.

3 p-th variation and wavelet coefficients

The results obtained in [7], in particular [7, Proposition 4.2 and Theorem 4.3] provide conditions on the Schauder coefficients which characterize the boundedness of the (generalized) p-th variations along a given partition sequence π.

However, the Schauder basis used in [7] is constructed from the partition sequence π itself. Consequently, both the basis and the associated coefficients depend strongly on the choice of π. This makes it difficult to compare the behaviour of the p-th variation of a given path along different partition sequences. To overcome this difficulty, we now establish sufficient conditions for the boundedness of generalized p-th variation in terms of wavelet coefficients. Since the wavelet basis is fixed independently of the partition sequence, these coefficients provide a common multiscale representation of the path.

3.1 Wavelet criteria

In this section, we prove an upper bound of generalized p-th variation along any balanced sequence of partitions π with a weighted L^p norm of wavelet coefficients. We would like to remark that in the following theorem, only the maximum step size of the partition, i.e. $|\pi^n|$ plays a role in the upper bound, and this can be replaced by the step size of dyadic partition

at an appropriate level. This suggests generalized p-th variation along any balanced sequence of partitions with mesh going to zero at a similar speed can be upper bounded by a single weighted L^p norm of wavelet coefficients (see Remark 3.5 for details).

Theorem 3.1. *Let $p > 1$ and let $\pi = (\pi^n)_{n\geq 0}$ be a balanced sequence of partitions of $[0,1]$ with vanishing mesh. Let $x \in C^\alpha([0,1])$ for some $0 < \alpha < 1$. Let ψ be a compactly supported wavelet of class C^1, with at least one vanishing moment. Fix*

$$0 < \beta < \min\{1, p\alpha\},$$

and define

$$J_\beta^\pi(n) := \left\lfloor \frac{1}{\beta} \log_2 \left(\frac{1}{|\pi^n|} \right) \right\rfloor \in \mathbb{N}. \tag{11}$$

Recall $A_j^{(p)}$ is defined in (10). If one sets

$$\eta_{n,\beta}^{\pi,(p)} = |\pi^n|^{p-1} \left(\sum_{j=0}^{J_\beta^\pi(n)} 2^{j(1-\frac{1}{p})} A_j^{(p)} \right)^p \tag{12}$$

then

$$\limsup_{n\to\infty} [x]_{\pi^n}^{(p)}(1) \lesssim \limsup_{n\to\infty} \eta_{n,\beta}^{\pi,(p)}.$$

In particular,

$$\limsup_{n\to\infty} \eta_{n,\beta}^{\pi,(p)} < \infty \quad \Longrightarrow \quad x \in \mathcal{X}_\pi^p.$$

Remark 3.2. *By the definition of $J_\beta^\pi(n)$ in (11), we have*

$$2^{-\beta(J_\beta^\pi(n)+1)} < |\pi^n| \leq 2^{-\beta J_\beta^\pi(n)}.$$

Consequently, equation (12) can be rewritten as

$$\eta_{n,\beta}^{\pi,(p)} \asymp \left(\sum_{j=0}^{J_\beta^\pi(n)} 2^{(j-\beta J_\beta^\pi(n))(1-\frac{1}{p})} A_j^{(p)} \right)^p.$$

Note that $\eta_{n,\beta}^{\pi,(p)}$ depends on the partition π^n only through its mesh size $|\pi^n|$, and not through the positions of its partition points.

Remark 3.3. *Assume in addition that*

$$x \in \bigcap_{0<\varepsilon<1/p} C^{1/p-\varepsilon}([0,1]).$$

Then the conclusion of Theorem 3.1 holds for every $\beta \in (0,1)$. For every fixed n, the quantity $\eta_{n,\beta}^{\pi,(p)}$ is non-increasing with respect to β. Moreover, the implicit constant in the estimate

$$\limsup_{n\to\infty} [x]_{\pi^n}^{(p)} \lesssim \limsup_{n\to\infty} \eta_{n,\beta}^{\pi,(p)}$$

is independent of β. Therefore,

$$\limsup_{n\to\infty} [x]_{\pi^n}^{(p)} \lesssim \inf_{0<\beta<1} \limsup_{n\to\infty} \eta_{n,\beta}^{\pi,(p)} = \lim_{\beta\uparrow 1} \limsup_{n\to\infty} \eta_{n,\beta}^{\pi,(p)}.$$

Since the wavelet coefficients are independent of the partition sequence, it is natural to compare the corresponding criteria for different partition sequences. The following proposition shows that their boundedness is preserved whenever the associated sequences of scales remain sufficiently close.

Proposition 3.4. *Fix $p > 1$, $\beta \in (0, 1]$ and $x \in C^0([0, 1])$, and let $\pi = (\pi^n)_{n\geq 0}$ and $\sigma = (\sigma^n)_{n\geq 0}$ be two partition sequences with vanishing mesh. Define*

$$\mathcal{J}_\beta^\pi := \{J_\beta^\pi(n) : n \geq 0\}, \qquad \mathcal{J}_\beta^\sigma := \{J_\beta^\sigma(m) : m \geq 0\}.$$

Assume that there exists $K \geq 0$ such that, for every $J \in \mathcal{J}_\beta^\pi$, there exists $J' \in \mathcal{J}_\beta^\sigma$ satisfying

$$J \leq J' \leq J + K,$$

and conversely, with π and σ interchanged. Then

$$\sup_n \eta_{n,\beta}^{\pi,(p)} < \infty \qquad \Longleftrightarrow \qquad \sup_m \eta_{m,\beta}^{\sigma,(p)} < \infty.$$

Proof. Let us set

$$F_\beta(J) = 2^{-\beta(1-\frac{1}{p})J} \sum_{j=0}^{J} 2^{j(1-\frac{1}{p})} A_j^{(p)}, \quad J \geq 0.$$

By the definition of $J_\beta^\pi(n)$, one has $|\pi^n|^{p-1} \asymp 2^{-\beta J_\beta^\pi(n)(p-1)}$, with constants independent of n. Therefore,

$$\left(\eta_{n,\beta}^{\pi,(p)}\right)^{1/p} \asymp F_\beta(J_\beta^\pi(n))$$

and similarly,

$$\left(\eta_{m,\beta}^{\sigma,(p)}\right)^{1/p} \asymp F_\beta(J_\beta^\sigma(m)).$$

Assume that $M_\sigma := \sup_{m\geq 0} F_\beta\big(J_\beta^\sigma(m)\big) < \infty$. Let $n \geq 0$ and set $J := J_\beta^\pi(n)$. By assumption, there exists $m \geq 0$ such that

$$J \leq J' := J_\beta^\sigma(m) \leq J + K.$$

Since all the terms in the sum defining F_β are non-negative, we have

$$\sum_{j=0}^{J} 2^{j(1-\frac{1}{p})} A_j^{(p)} \leq \sum_{j=0}^{J'} 2^{j(1-\frac{1}{p})} A_j^{(p)}.$$

Moreover,

$$2^{-\beta(1-\frac{1}{p})J} = 2^{\beta(1-\frac{1}{p})(J'-J)} 2^{-\beta(1-\frac{1}{p})J'} \leq 2^{\beta(1-\frac{1}{p})K} 2^{-\beta(1-\frac{1}{p})J'}.$$

Therefore,

$$F_\beta(J) \leq 2^{\beta(1-\frac{1}{p})K} F_\beta(J') \leq 2^{\beta(1-\frac{1}{p})K} M_\sigma.$$

Since n was arbitrary, it follows that

$$\sup_{n\geq 0} F_\beta\big(J_\beta^\pi(n)\big) \leq 2^{\beta(1-\frac{1}{p})K} \sup_{m\geq 0} F_\beta\big(J_\beta^\sigma(m)\big).$$

Applying the same argument with π and σ interchanged gives the conclusion. □

Remark 3.5. *Proposition 3.4 shows that the boundedness of the wavelet criterion only depends on which dyadic scales are reached by the partition sequence, provided their scale sets satisfy the two-sided bounded-shift assumption of Proposition 3.4. In particular, if π and σ are balanced and their scale sets satisfy this assumption, and if x satisfies the assumptions of Theorem 3.1, then*

$$\sup_n \eta_{n,\beta}^{\pi,(p)} < \infty \qquad \Longleftrightarrow \qquad \sup_m \eta_{m,\beta}^{\sigma,(p)} < \infty,$$

and either of these equivalent conditions implies

$$x \in \mathcal{X}_\pi^p \cap \mathcal{X}_\sigma^p.$$

In other words, there are partition-independent wavelet conditions that guarantee membership of $\mathcal{X}_\pi^p$ for all balanced partitions π simultaneously. For instance, the condition

$$\sup_{J\geq 0} 2^{-\beta(1-1/p)J} \sum_{j=0}^{J} 2^{j(1-1/p)} A_j^{(p)} < \infty$$

does not involve any partition sequence and implies boundedness of the corresponding wavelet criterion along every sequence with vanishing mesh. However, we note that the partition-independent wavelet condition is sufficient, not necessary.

3.2 Besov space Vs generalized p-th variation space

Let us now give a Besov interpretation of the wavelet criterion, starting with the dyadic partition sequence. Recall that the wavelet coefficients introduced in (9) are those of the zero extension of $y = x - \tilde{x}$ to $\mathbb{R}$. Throughout the paper, unless specified explicitly, Besov spaces appearing in the statements are understood on $[0,1]$. The wavelet characterization below is first applied to the zero extension on $\mathbb{R}$; the relation between these two domains will be specified at the critical indices. It is well-known that the wavelet coefficients $(C_k)_{k\in\mathbb{Z}}$ and $(c_{j,k})_{j,k}$ characterize Besov regularity. More precisely, for $1 \leq p, q \leq \infty$ and $0 < s < 1$, we have $y \in B_{p,q}^s(\mathbb{R})$ if and only if

$$\Big(\sum_{k\in\mathbb{Z}} |C_k|^p\Big)^{1/p} < \infty \qquad \text{and} \qquad \Big(2^{j(s-\frac{1}{p})} A_j^{(p)}\Big)_{j\geq 0} \in \ell^q(\mathbb{N}),$$

where $A_j^{(p)}$ is defined in (10); see [14, Chapter 6, Section 10, (10.1)–(10.2)], with the usual modifications when $p = \infty$ or $q = \infty$.

Since y and φ are compactly supported, only finitely many coefficients C_k are nonzero, and the first condition is automatically satisfied.

In particular, at the critical smoothness $s = \frac{1}{p}$,

$$y \in B_{p,\infty}^{1/p}(\mathbb{R}) \iff \sup_{j\geq 0} A_j^{(p)} < \infty, \qquad y \in B_{p,1}^{1/p}(\mathbb{R}) \iff \sum_{j\geq 0} A_j^{(p)} < \infty. \tag{13}$$

Since $\tilde{x}$ is affine, it belongs to $B_{p,q}^s([0,1])$ for every $0 < s < 1$. As Besov spaces are vector spaces, we have

$$x \in B_{p,q}^s([0,1]) \quad \Longleftrightarrow \quad y = x - \tilde{x} \in B_{p,q}^s([0,1]).$$

For $1 < p < \infty$ and $1/p \leq s < 1$, the endpoint condition $y(0) = y(1) = 0$ gives, for continuous x,

$$x \in B_{p,\infty}^s([0,1]) \quad \Longleftrightarrow \quad y \in B_{p,\infty}^s(\mathbb{R}),$$

where on the right y denotes its zero extension. Indeed, the intrinsic difference-quotient characterization controls increments inside $[0,1]$. For increments crossing the endpoints, one uses $\|y\|_{L^p(0,h)} \lesssim h^s$ and the analogous estimate near 1. These bounds follow from boundedness

when $s = 1/p$, and from $B^s_{p,\infty}([0,1]) \hookrightarrow C^{s-1/p}([0,1])$ and the vanishing endpoint values when $s > 1/p$.

At the critical summability index $q = 1$, however, vanishing endpoint values alone do not ensure that the zero extension belongs to $B^{1/p}_{p,1}(\mathbb{R})$. Hence the second equivalence in (13) is used only on $\mathbb{R}$; in particular,

$$\sum_{j\geq 0} A_j^{(p)} < \infty \quad \Longrightarrow \quad x \in B^{1/p}_{p,1}([0,1]).$$

The following Lemma is a preliminary for the Besov interpretation in Proposition 3.7.

Lemma 3.6. *Let $\beta \in (0,1]$, $\theta > 0$, and let $(a_j)_{j\geq 0}$ be a sequence of non-negative numbers. Then*

$$\sup_{J\geq 0} 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\theta j} a_j < \infty \qquad \Longleftrightarrow \qquad \sup_{J\geq 0} 2^{(1-\beta)\theta J} a_J < \infty.$$

Proof. Define

$$F_\beta(J) := 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\theta j} a_j, \qquad J \geq 0.$$

The first implication is immediate: since all terms are non-negative, one has

$$F_\beta(J) = 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\theta j} a_j \geq 2^{-\beta\theta J} 2^{\theta J} a_J = 2^{(1-\beta)\theta J} a_J.$$

Conversely, assume that

$$M := \sup_{j\geq 0} 2^{(1-\beta)\theta j} a_j < \infty.$$

Then,

$$F_\beta(J) = 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\theta j} a_j \leq M\, 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\theta j} 2^{-(1-\beta)\theta j} = M\, 2^{-\beta\theta J} \sum_{j=0}^{J} 2^{\beta\theta j} \leq M\, \frac{2^{\beta\theta}}{2^{\beta\theta} - 1},$$

uniformly in J. This proves the converse implication. □

Proposition 3.7. *Let $p > 1$ and $\beta \in (0,1]$ and $x \in C^0([0,1],\mathbb{R})$. Let $\mathbb{T} = (\mathbb{T}^n)_{n\geq 0}$ denote the dyadic partition sequence of $[0,1]$ and define*

$$s_\beta := 1 - \beta\left(1 - \frac{1}{p}\right).$$

Then

$$\sup_{n\geq 0} \eta^{\mathbb{T},(p)}_{n,\beta} < \infty \quad \Longleftrightarrow \quad x \in B^{s_\beta}_{p,\infty}([0,1]).$$

If the sequence of partition π with vanishing mesh has uniform bounded gap (3), then there exists $K \geq 0$ such that

$$J^\pi_\beta(n+1) - J^\pi_\beta(n) \leq K, \quad n \geq 0.$$

Since $\pi = (\pi^n)_{n\geq 0}$ has vanishing mesh $(J^\pi_\beta(n))_{n\geq 0}$ also tends to infinity. Consequently, there exists $K_\pi > 0$ such that, for every integer $J \geq 0$, one can find n satisfying

$$J \leq J^\pi_\beta(n) \leq J + K_\pi.$$

The same property holds for the dyadic sequence $(J^{\mathbb{T}}_\beta(n))_{n\geq 0}$ as well. Hence the two scale sequences satisfy the assumptions of Proposition 3.4. We therefore obtain the following corollary.

Corollary 3.8. *Let $p > 1$, $\beta \in (0, 1]$, and let $\pi = (\pi^n)_{n\geq 0}$ be a partition sequence with uniform bounded gap and vanishing mesh. Then*

$$\sup_{n\geq 0} \eta_{n,\beta}^{\pi,(p)} < \infty \quad \Longleftrightarrow \quad x \in B_{p,\infty}^{s_\beta}([0, 1]),$$

where

$$s_\beta = 1 - \beta\left(1 - \frac{1}{p}\right).$$

Corollary 3.8 shows that, within the class of partition sequences with uniform bounded gap, the boundedness of the wavelet criterion is independent of the particular choice of partition sequence.

Indeed, if π and σ are two such partition sequences, then

$$\sup_n \eta_{n,\beta}^{\pi,(p)} < \infty \qquad \Longleftrightarrow \qquad x \in B_{p,\infty}^{s_\beta}([0, 1]) \qquad \Longleftrightarrow \qquad \sup_m \eta_{m,\beta}^{\sigma,(p)} < \infty.$$

Thus, in this class, the wavelet criterion depends only on the Besov regularity of the path and not on the precise location of the partition points.

Remark 3.9. *At the critical value $\beta = 1$, one has*

$$s_1 = \frac{1}{p},$$

and therefore

$$\sup_n \eta_{n,1}^{\pi,(p)} < \infty \qquad \Longleftrightarrow \qquad x \in B_{p,\infty}^{1/p},$$

for every partition sequence satisfying the assumptions of Corollary 3.8.

For every $\beta < 1$, one has

$$s_\beta > \frac{1}{p}.$$

Hence the sufficient condition appearing in Theorem 3.1 requires a Besov regularity strictly stronger than the critical exponent $1/p$.

The following result shows that the converse implication in Theorem 3.1 is false in general.

Proposition 3.10. *Let $p \geq 2$ and let B_H be a fractional Brownian motion with Hurst parameter $H \in (1/p, 1)$. Set*

$$X(t) = B_H(t) - tB_H(1), \qquad t \in [0, 1],$$

and let $\mathbb{T} = (\mathbb{T}^n)_{n\geq 0}$ be the dyadic partition sequence. Then, almost surely,

$$X \in \mathcal{X}_{\mathbb{T}}^p.$$

Moreover, for every

$$0 < \beta < \frac{1 - H}{1 - \frac{1}{p}},$$

one has

$$\sup_{n\geq 0} \eta_{n,\beta}^{\mathbb{T},(p)} = +\infty \quad a.s.$$

Proof. Since $H > 1/p$, we may choose $\alpha \in (1/p, H)$. Almost surely, the sample paths of B_H belong to $C^\alpha([0, 1])$. The same is therefore true for X. Consequently,

$$[X]_{\mathbb{T}^n}^{(p)}(1) \leq C(\omega)^p \sum_{k=0}^{2^n-1} 2^{-\alpha p n} = C(\omega)^p 2^{-n(\alpha p-1)}.$$

Since $\alpha p > 1$, it follows that

$$X \in \mathcal{X}_{\mathbb{T}}^{p} \qquad \text{a.s.}$$

We now study the wavelet criterion. Let us fix

$$0 < \beta < \frac{1-H}{1-\frac{1}{p}}.$$

Then, by definition of s_β, one has

$$s_\beta = 1 - \beta\left(1 - \frac{1}{p}\right) > H.$$

Using Proposition 2 of Rosenbaum [16] applied with $r = H$ and $\alpha = 1$ (since the increment standard deviation of B_H is $\sigma(h) = h^H$), since $p \geq 2$ and $Hp > 1$, we obtain

$$B_H \notin B_{p,\infty}^{s_\beta} \qquad \text{a.s.}$$

On the other hand,

$$t \longmapsto tB_H(1)$$

is affine and therefore belongs to $B_{p,\infty}^{s_\beta}([0,1])$, since $s_\beta < 1$. Hence

$$X = B_H - tB_H(1) \notin B_{p,\infty}^{s_\beta}([0,1]) \qquad \text{a.s.}$$

Proposition 3.7 implies then that

$$\sup_{n\geq 0} \eta_{n,\beta}^{\mathbb{T},(p)} = +\infty \qquad \text{a.s.}$$

□

4 The reversed inclusion

In [7, Proposition 4.2 and Theorem 4.3], the authors characterized membership in the generalized p-th variation class in terms of ℓ^p-norms of partition-dependent Schauder coefficients. The wavelet criterion considered here, however, does not yield such a characterization without additional assumptions. Proposition 3.10 shows that the implication of Theorem 3.1 cannot be reversed: for fractional Brownian motion with $H > 1/p$ the path lies in $\mathcal{X}_{\mathbb{T}}^{p}$, while $\sup_n \eta_{n,\beta}^{\mathbb{T},(p)} = \infty$ for all sufficiently small β. The reason is transparent from Remark 3.9: for $\beta < 1$ the finiteness of $\eta_{n,\beta}^{\pi,(p)}$ is a Besov condition at smoothness $s_\beta > 1/p$, strictly above the critical index $1/p$, and finite p-th variation does not force any regularity above the critical index. The only candidate for a converse is therefore the endpoint $\beta = 1$, where $s_1 = 1/p$. In this section we show that at the endpoint the converse does hold. For balanced complete refining sequences, the p-th variation π control the quantity $\eta_n^{\pi,(p)} := \eta_{n,1}^{\pi,(p)}$, and as a consequence the partition independent ℓ^p type norm of Wavelet coefficients $A_j^{(p)}$.

Theorem 4.1. *Let $p > 1$ and let $\pi = (\pi^n)_{n\geq 0}$ be a balanced and complete refining sequence of partitions of $[0,1]$ with vanishing mesh. Then there exists a constant $C > 0$ such that, for every $x \in C^0([0,1])$,*

$$\limsup_{n\to\infty} \eta_n^{\pi,(p)} \leq C \limsup_{n\to\infty} [x]_{\pi^n}^{(p)}(1).$$

In particular,

$$\mathcal{X}_{\pi}^{p} \subset B_{p,\infty}^{1/p}([0,1]).$$

Proof. The first assertion is a consequence of Lemma 4.2 and Proposition 4.3 described below. Now let $x \in \mathcal{X}_\pi^p$. By definition, $\limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(1) < \infty$ and the first assertion therefore implies that $\sup_n \eta_n^{\pi,(p)} < \infty$. Since every complete refining partition sequence has uniformly bounded gaps, we can apply Corollary 3.8 with $\beta = 1$ to obtain $x \in B^{1/p}_{p,\infty}([0,1])$. □

Before proving Theorem 4.1, let us first introduce/recall few notations. For the case $\beta = 1$ for notational convenience we will write $J^\pi := J_1^\pi$, i.e.

$$J^\pi(n) := J_1^\pi(n) = \left\lfloor \log_2\left(\frac{1}{|\pi^n|}\right)\right\rfloor \in \mathbb{N}. \tag{14}$$

Define corresponding to (12) for $\beta = 1$,

$$\eta_n^{\pi,(p)} := |\pi^n|^{p-1}\Big(\sum_{j=0}^{J^\pi(n)} 2^{j\theta} A_j^{(p)}\Big)^p,$$

where $A_j^{(p)}$, the partition free wavelet contribution, is defined in (10). We will now show that $\eta_n^{\pi,(p)}$ is bounded above by $A_j^{(p)}$ (Lemma 4.2) and then $A_j^{(p)}$'s are bounded above by generalized p-th variation along π (Proposition 4.3).

Lemma 4.2. *Let $p > 1$, $\theta := 1 - \frac{1}{p}$, and let π be a sequence of partitions of $[0,1]$ with vanishing mesh. Let $J^\pi(n)$ as in (14) and $A_j^{(p)}$ as in (10). Then*

$$\limsup_{n\to\infty} \eta_n^{\pi,(p)} \;\le\; \frac{1}{(1-2^{-\theta})^p}\;\limsup_{j\to\infty}(A_j^{(p)})^p.$$

Proof of Lemma 4.2. Since (14) implies $|\pi^n| \le 2^{-J^\pi(n)}$, we have $|\pi^n|^\theta 2^{J^\pi(n)\theta} \le 1$ and hence

$$\big(\eta_n^{\pi,(p)}\big)^{1/p} = |\pi^n|^\theta \sum_{j=0}^{J^\pi(n)} 2^{j\theta} A_j^{(p)} \le \sum_{j=0}^{J^\pi(n)} 2^{-(J^\pi(n)-j)\theta} A_j^{(p)} = \frac{1}{1-2^{-\theta}} \sum_{j=0}^{\infty} a_{n,j} A_j^{(p)},$$

where

$$a_{n,j} := \begin{cases} (1-2^{-\theta})\, 2^{-(J^\pi(n)-j)\theta}, & 0 \le j \le J^\pi(n), \\ 0, & j > J^\pi(n). \end{cases}$$

Now we apply (generalized) Silverman–Toeplitz theorem [7, Lemma 4.8] to conclude the proof. □

Proposition 4.3. *Consider $p > 1$ and $x \in C^0([0,1])$. Let π be a balanced and complete refining sequence of partitions of $[0,1]$. Denote,*

$$\rho := (1+a)^{-\min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}} \in (0,1) \;\; \text{and} \;\; n(j) := \max\{m : |\pi^m| \ge 2^{-j}\}. \tag{15}$$

Then there is a constant C_1, independent of j and x, such that for every $j \ge 0$

$$A_j^{(p)} \;\le\; C_1 \sum_{m\ge 0} \rho^{\,|m-n(j)|}\Big([x]^{(p)}_{\pi^{m+1}}(1)\Big)^{1/p}. \tag{16}$$

In particular

$$\limsup_{j\to\infty}\Big(A_j^{(p)}\Big)^p \;\le\; \Big(\frac{2C_1}{1-\rho}\Big)^p \limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(1). \tag{17}$$

5 Invariance of Besov, p-variation and p-th variation index

The generalized p-th variation of a continuous path may depend strongly on the chosen sequence of partitions. In particular, membership in $\mathcal{X}_\pi^p$ need not be preserved when the partition sequence is changed. However, the critical exponent separating bounded and unbounded variation may exhibit a much stronger form of stability.

The results of the preceding sections provide a natural way to investigate this phenomenon through Besov regularity. On the one hand, Theorem 4.1 shows that, for balanced and complete refining partition sequences,

$$\mathcal{X}_\pi^p \subset B_{p,\infty}^{1/p}([0,1]).$$

On the other hand, the classical embedding of $B_{p,1}^{1/p}([0,1])$ into the space of paths with finite p variation yields

$$B_{p,1}^{1/p}([0,1]) \subset \mathcal{V}_{p\text{-var}} \subset \mathcal{X}_\pi^p.$$

These inclusions reveal a close connection between generalized p-th variation and Besov regularity at the critical smoothness $1/p$. Although these Besov spaces are different, we show that they determine the same critical index for paths with positive Hölder regularity.

More precisely, we identify the critical Besov index with both the classical variation index and the variation index along any balanced refining partition sequence with vanishing mesh and bounded gap. The equality between the latter two indices was established in [13]; our contribution is to provide an intrinsic characterization of their common value through Besov regularity and wavelet coefficients.

Proposition 5.1. *Let $p > 1$ and let σ be a balanced sequence of partitions of $[0,1]$ with vanishing mesh, and let $x \in C^0([0,1])$.*

(i) If the following partition independent condition holds

$$\sum_{j\geq 0} A_j^{(p)} < \infty,$$

then $x \in V_\sigma^p$ with $[x]_\sigma^{(p)} \equiv 0$; in particular $x \in \mathcal{X}_\sigma^p$.

(ii) Let π be a balanced and complete refining sequence of partitions of $[0,1]$ with vanishing mesh. If $x \in \mathcal{X}_\pi^p \cap C^\alpha([0,1])$ for some $\alpha > 0$, then

$$\sum_{j\geq 0} A_j^{(r)} < \infty, \qquad \forall r > p.$$

Consequently,

$$x \in \mathcal{X}_\pi^p \quad \Longrightarrow \quad x \in V_\sigma^r \ \text{ with } [x]_\sigma^{(r)} \equiv 0 \ \text{ for every } r > p.$$

So combining Theorem 3.1, Corollary 3.8, Theorem 4.1 and Proposition 5.1 we get the following inclusion of Besov spaces.

Corollary 5.2. *Let $p > 1$ and let π be a balanced and complete refining sequence of partitions of $[0,1]$. Then for every $s > \frac{1}{p}$,*

$$B_{p,\infty}^s([0,1]) \subset B_{p,1}^{1/p}([0,1]) \ \subset \ V_\pi^p \ \subset \ \mathcal{X}_\pi^p \ \subset \ B_{p,\infty}^{1/p}([0,1]), \tag{18}$$

and every $x \in B_{p,1}^{1/p}([0,1])$ satisfies $[x]_\pi^{(p)} \equiv 0$. The second inclusion and the last assertion hold for every balanced sequence π with vanishing mesh. The first inclusion is well-known in wavelet theory [14, Chapter 6]. For the second inclusion and the vanishing p-th variation, we

use the classical embedding $B^{1/p}_{p,1}([0,1]) \hookrightarrow \mathcal{V}_{p-var}$ (see [2]) together with the density of smooth functions in $B^{1/p}_{p,1}([0,1])$. Indeed, by continuity of this embedding, the classical p variation of the approximation error tends to zero, while smooth functions have vanishing discrete p-th variation along any partition sequence with vanishing mesh. In particular, these conclusions do not require zero extension in $B^{1/p}_{p,1}(\mathbb{R})$.

Remark 5.3. *We would also like to highlight that such inclusions of Besov spaces and classical p-variation spaces have already been well explored in the literature [2]. Combining with* (18) *we get,*

$$B^{1/p}_{p,1}([0,1]) \;\subset\; \mathcal{V}_{p-var} \;\subset\; \mathcal{X}^p_\pi \;\subset\; B^{1/p}_{p,\infty}([0,1]). \tag{19}$$

Note that V^p_π and $\mathcal{V}_{p\text{-}var}$ have different definitions: the former requires convergence of the discrete p-th variations along one prescribed sequence, while the latter requires a uniform bound over all finite partitions.

The preceding inclusions suggest that, although generalized p-th variation may depend on the partition sequence, its critical exponent can be described in terms of intrinsic Besov regularity. For $x \in C^0([0,1])$, recall the variation index along π,

$$p^\pi(x) := \inf\left\{p \geq 1 : \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(1) = 0\right\},$$

and define the classical variation index by

$$p_{\mathrm{var}}(x) := \inf\left\{p \geq 1 : \|x\|_{p\text{-var}} < \infty\right\}.$$

Similarly, for $1 \leq q \leq \infty$, define the critical Besov index

$$p_{\mathrm{Besov},q}(x) := \inf\left\{p > 1 : x \in B^{1/p}_{p,q}([0,1])\right\}.$$

As usual, we adopt the convention $\inf \emptyset = \infty$. Our main result identifies these indices for paths with positive Hölder regularity.

Theorem 5.4. *Let π be a balanced, refining sequence of partitions of $[0,1]$ with vanishing mesh and bounded gap* (3)*. Then for every $x \in C^0([0,1])$,*

$$p_{\mathrm{Besov},\infty}(x) \;\leq\; p^\pi(x) \;\leq\; p_{\mathrm{var}}(x) \;\leq\; p_{\mathrm{Besov},1}(x). \tag{20}$$

If moreover $x \in C^\alpha([0,1])$ for some $\alpha > 0$, then all these indices coincide:

$$p^\pi(x) = p_{\mathrm{var}}(x) = p_{\mathrm{Besov},q}(x), \qquad 1 \leq q \leq \infty.$$

In particular, for such x, the variation index $p^\pi(x)$ is the same for all partition sequences π in this class and is determined fully by the wavelet coefficients of x.

Proof of Theorem 5.4. Since $|\pi^n| \downarrow 0$, there is a subsequence $\tau = (\pi^{\ell_n})_n$ of π which is balanced and complete refining. One possible way of defining such ℓ_n is simply,

$$\ell_0 := 0, \qquad \ell_{n+1} := \min\{m > \ell_n \;\big|\; |\pi^m| \leq \tfrac{1}{1+a}|\pi^{\ell_n}|\},$$

for any choice of $a > 0$. Indeed, if K is the bounded gap constant, then $1 + a \leq |\tau^n|/|\tau^{n+1}| \leq K(1+a)$. Balance is inherited from π, and these two bounds imply that the number of intervals of τ^{n+1} contained in any interval of τ^n is uniformly bounded. Hence τ is complete refining. Since τ is a subsequence of π, we have, for every $q > 1$,

$$x \in \mathcal{X}^q_\pi \implies x \in \mathcal{X}^q_\tau.$$

Moreover, vanishing variations remain vanishing along a subsequence, so $p^\tau(x) \le p^\pi(x)$. Since τ is balanced and complete refining, Remark 5.3 gives, for every $q > 1$,

$$B^{1/q}_{q,1}([0,1]) \;\subset\; \mathcal{V}_{q-var} \;\subset\; \mathcal{X}^q_\tau \;\subset\; B^{1/q}_{q,\infty}([0,1]).$$

Moreover $\mathcal{V}_{q-var} \subset \mathcal{X}^q_\pi$, as $[x]^{(q)}_{\pi^n}(1) \le \|x\|^q_{q\text{-var}}$ for every n. For $r > q$, bounded classical q-variation also yields $[x]^{(r)}_{\pi^n}(1) \le \omega_x(|\pi^n|)^{r-q}\|x\|^q_{q-var} \to 0$, so $p^\pi(x) \le p_{\mathrm{var}}(x)$. Taking the infimum over q we get,

$$p_{\mathrm{Besov},\infty}(x) \;\le\; p^\tau(x) \;\le\; p^\pi(x) \;\le\; p_{\mathrm{var}}(x) \;\le\; p_{\mathrm{Besov},1}(x),$$

which proves (20).

Now let $x \in C^\alpha([0,1])$ for some $\alpha > 0$. Fix $0 < \alpha_0 < \min\{\alpha, 1\}$. Then $p_{\mathrm{Besov},\infty}(x) < \infty$, since $x \in B^{1/q}_{q,\infty}([0,1])$ whenever $q > 1$ and $1/q < \alpha$. Let $q > p_{\mathrm{Besov},\infty}(x)$. By the definition of the infimum and the monotonicity of the ℓ^q norms, we can choose q_0 with $p_{\mathrm{Besov},\infty}(x) < q_0 < q$ and $x \in B^{1/q_0}_{q_0,\infty}([0,1])$. Then $\sup_j A^{(q_0)}_j < \infty$ and since $A^{(q)}_j \le A^{(q_0)}_j$, we also have $\sup_j A^{(q)}_j < \infty$. So, for all $r > q$ we first use Hölder inequality and then use the bound (8) to get,

$$\big(A^{(r)}_j\big)^r \le \Big(\sup_k |c_{j,k}|\Big)^{r-q}\big(A^{(q)}_j\big)^q \lessapprox \Big(\sup_{i\ge 0} A^{(q)}_i\Big)^q 2^{-\alpha_0(r-q)j}.$$

So $\sum_{j\ge 0} A^{(r)}_j < \infty$. As a consequence of (13) we have $y \in B^{1/r}_{r,1}(\mathbb{R})$, and by restriction and the smoothness of $\tilde{x}$, one then has $x \in B^{1/r}_{r,1}([0,1])$ and hence $p_{\mathrm{Besov},1}(x) \le r$. Letting $r \downarrow q \downarrow p_{\mathrm{Besov},\infty}(x)$ we get $p_{\mathrm{Besov},1}(x) \le p_{\mathrm{Besov},\infty}(x)$, so all the inequalities above are equalities. Since $B^{1/r}_{r,1}([0,1]) \subset B^{1/r}_{r,q}([0,1]) \subset B^{1/r}_{r,\infty}([0,1])$ for every $1 \le q \le \infty$, we also have

$$p_{\mathrm{Besov},\infty}(x) \le p_{\mathrm{Besov},q}(x) \le p_{\mathrm{Besov},1}(x).$$

Since $p_{\mathrm{Besov},\infty}(x)$ does not depend on π, neither does $p^\pi(x)$, concluding the claim. □

Remark 5.5. *For paths with positive Hölder regularity, Theorem 5.4 recovers the equality between the variation index along π and the classical variation index established in [13, Theorem 2.1]. Our result complements this equality by identifying their common value with the critical Besov indices on $[0,1]$, via wavelet coefficients independent of the partition sequence.*

6 A pathwise change of variable formula in $B^{1/p}_{p,\infty}$

Föllmer's pathwise Itô formula and its higher-order versions [5, 6] are obtained for $x \in V^p_\pi$ along a fixed sequence of partitions π: they require the discrete p-th variations of x to converge along π, and both the pathwise integral and the correction term are defined as limits along π. In general these limits heavily depend on the choice of π [11], so that these results do not allow one to compare the formulas obtained along two different sequences of partitions. In fact for Hölder contiunous paths in $B^{1/p}_{p,\infty}([0,1])$, the p-th variation need not exist along any sequence. We nevertheless show that a change of variable formula holds at every order $r > p$ i.e. the compensated Riemann sums of order r converge along every sequence of partitions with vanishing mesh, and to the same limit. The resulting pathwise integral is therefore independent of the sequence of partitions. This is in the spirit of rough path theory, where the integral is a limit of compensated Riemann sums along arbitrary partitions with vanishing mesh [12]; here the order of the compensation is determined by the critical Besov exponent p, and not by the Hölder exponent or p-variation of x.

For $m \in \mathbb{N}$, $\gamma \in (0,1]$ and a compact interval $K \subset \mathbb{R}$, we denote the γ-Hölder constant of $g : \mathbb{R} \to \mathbb{R}$ on K by

$$\|g\|_{C^\gamma;K} := \sup_{u \neq v \in K} \frac{|g(u) - g(v)|}{|u - v|^\gamma},$$

and by $C^{m,\gamma}_{\mathrm{loc}}(\mathbb{R})$ the space of functions $f \in C^m(\mathbb{R})$ such that $\|f^{(m)}\|_{C^\gamma;K} < \infty$ for every compact interval K.

Theorem 6.1. *Let $p > 1$ and $x \in B^{1/p}_{p,\infty}([0,1]) \cap C^\alpha([0,1])$ for some $\alpha > 0$. Let $r > p$, and set $m_r := \lceil r \rceil - 1$ and $\gamma_r := r - m_r \in (0,1]$. Then, for every sequence of partitions $\sigma = (\sigma^n)_{n \geq 0}$ of $[0,1]$ with vanishing mesh, $\sigma^n = \{0 = s^n_0 < \cdots < s^n_{N(\sigma^n)} = 1\}$, and every $f \in C^{m_r,\gamma_r}_{\mathrm{loc}}(\mathbb{R})$,*

$$f\big(x(t)\big) - f\big(x(0)\big) = \lim_{n \to \infty} \sum_{\ell=0}^{N(\sigma^n)-1} \mathbf{1}_{\{s^n_{\ell+1} \leq t\}} \sum_{k=1}^{m_r} \frac{f^{(k)}\big(x(s^n_\ell)\big)}{k!} \big(x(s^n_{\ell+1}) - x(s^n_\ell)\big)^k, \tag{21}$$

uniformly in $t \in [0,1]$. In particular, the limit does not depend on the sequence of partitions.

Proof. We will first show that the r-th variation vanishes for any $r > p$. To show this first fix $p < r_0 < r$ and $0 < \alpha_0 < \min\{\alpha, 1\}$. Since $x \in B^{1/p}_{p,\infty}([0,1])$, we have $\sup_{j \geq 0} A^{(p)}_j < \infty$ by (13). As in the proof of Theorem 5.4, Hölder's inequality and the bound (8) give

$$\big(A^{(r_0)}_j\big)^{r_0} \leq \Big(\sup_k |c_{j,k}|\Big)^{r_0 - p} \big(A^{(p)}_j\big)^p \lessapprox \Big(\sup_{i \geq 0} A^{(p)}_i\Big)^p 2^{-\alpha_0(r_0 - p)j},$$

so that $\sum_{j \geq 0} A^{(r_0)}_j < \infty$ for all $r_0 > p$. By (13) and restriction, $x \in B^{1/r_0}_{r_0,1}([0,1])$, hence $\|x\|_{r_0-var} < \infty$ by the embedding $B^{1/r_0}_{r_0,1}([0,1]) \hookrightarrow \mathcal{V}_{r_0-var}$ of [2]. Therefore

$$[x]^{(r)}_{\sigma^n}(1) \leq \omega_x\big(|\sigma^n|\big)^{r - r_0} \|x\|^{r_0}_{r_0-var} \xrightarrow[n \to \infty]{} 0,$$

where ω_x denotes the modulus of continuity of x.

We will now use Taylor expansion to show the change of variable formula holds. Let $K := x([0,1])$ and, for $t \in [0,1]$, let $t_n := \max\{s \in \sigma^n : \ s \leq t\}$. Denote by $S_n(t)$ the sum on the right-hand side of (21). Applying Taylor's formula with integral remainder at order m_r on each interval of σ^n and summing, as in the proof of [6, Theorem 1.5], one obtains

$$\begin{aligned}
&\Big|f\big(x(t_n)\big) - f\big(x(0)\big) - S_n(t)\Big| \\
&\quad = \Big| \sum_{\ell=0}^{N(\sigma^n)-1} \mathbf{1}_{\{s^n_{\ell+1} \leq t\}} \frac{\big(x(s^n_{\ell+1}) - x(s^n_\ell)\big)^{m_r}}{(m_r - 1)!} \\
&\qquad\qquad \times \int_0^1 (1 - \lambda)^{m_r - 1} \Big(f^{(m_r)}\big(x(s^n_\ell) + \lambda\big(x(s^n_{\ell+1}) - x(s^n_\ell)\big)\big) - f^{(m_r)}\big(x(s^n_\ell)\big)\Big)\, d\lambda \Big| \\
&\quad \leq \frac{1}{m_r!} \|f^{(m_r)}\|_{C^{\gamma_r};K}\, [x]^{(r)}_{\sigma^n}(t) \\
&\quad \leq \frac{1}{m_r!} \|f^{(m_r)}\|_{C^{\gamma_r};K}\, [x]^{(r)}_{\sigma^n}(1),
\end{aligned}$$

which tends to zero uniformly in t by Step 1. Moreover $|t - t_n| \leq |\sigma^n| \to 0$ and $f \circ x$ is uniformly continuous, so $f(x(t_n)) \to f(x(t))$ uniformly in t. This proves (21). □

7 Proofs

In this section we will prove the main theorems of this manuscript.

7.1 Proof of Theorem 3.1

Proof of Theorem 3.1. We first estimate the contribution of all scales $j \le J_\beta^\pi(n)$. For every $j \ge 0$, let

$$X_j(t) = \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j t - k),$$

the contribution of the wavelet coefficients at level j. We have

$$X_j(t_{\ell+1}^n) - X_j(t_\ell^n) = \sum_{k\in K_{j,n,\ell}} c_{j,k}\left(\psi(2^j t_{\ell+1}^n - k) - \psi(2^j t_\ell^n - k)\right),$$

where we denote the effective index set $K_{j,n,\ell}$ as follows.

$$K_{j,n,\ell} := \left\{k \in \mathbb{Z} : \psi(2^j t_{\ell+1}^n - k) - \psi(2^j t_\ell^n - k) \neq 0\right\}.$$

Since $\mathrm{supp}(\psi) \subset [0,R]$, for every fixed $t \in [0,1]$, $\psi(2^j t - k) \neq 0 \implies 2^j t - k \in \mathrm{supp}(\psi) \subseteq [0,R]$. So there are at most $\lfloor R \rfloor + 1$ integers k such that $\psi(2^j t - k) \neq 0$. Since $k \in K_{j,n,\ell}$ implies that at least one of $\psi(2^j t_\ell^n - k)$ and $\psi(2^j t_{\ell+1}^n - k)$ is non-zero, we have

$$\#K_{j,n,\ell} \le 2(R+1) \qquad \text{for all } j,n,\ell.$$

Moreover, by the mean value theorem,

$$\left|\psi(2^j t_{\ell+1}^n - k) - \psi(2^j t_\ell^n - k)\right| \le \|\psi'\|_\infty 2^j |t_{\ell+1}^n - t_\ell^n|.$$

On the other hand, the triangle inequality, gives $\left|\psi(2^j t_{\ell+1}^n - k) - \psi(2^j t_\ell^n - k)\right| \le 2\|\psi\|_\infty$. Set

$$C_\psi := \max\left\{\|\psi'\|_\infty, 2\|\psi\|_\infty\right\}.$$

Combining the two previous estimates and using $|t_{\ell+1}^n - t_\ell^n| \le |\pi^n|$, we obtain

$$\left|\psi(2^j t_{\ell+1}^n - k) - \psi(2^j t_\ell^n - k)\right| \le C_\psi \min\left\{2^j |t_{\ell+1}^n - t_\ell^n|, 1\right\} \le C_\psi \min\left\{2^j |\pi^n|, 1\right\}.$$

Using Hölder's inequality and the uniform bound on the cardinality of $K_{j,n,\ell}$, we obtain

$$\begin{aligned}\left|X_j(t_{\ell+1}^n) - X_j(t_\ell^n)\right|^p &\le C_\psi^p \min\left\{2^j|\pi^n|,1\right\}^p \left(\#K_{j,n,\ell}\right)^{p-1} \sum_{k\in K_{j,n,\ell}} |c_{j,k}|^p \\ &\le C_\psi^p \big(2(R+1)\big)^{p-1} \min\left\{2^j|\pi^n|,1\right\}^p \sum_{k\in K_{j,n,\ell}} |c_{j,k}|^p.\end{aligned}$$

We now sum over the intervals of the finite partition π^n. Fix j and k, and denote

$$S_{j,k} = \mathrm{supp}(\psi(2^j \cdot - k)).$$

Since $\mathrm{supp}(\psi) \subset [0,R]$, the length of $S_{j,k}$ is bounded by $R2^{-j}$. If $k \in K_{j,n,\ell}$, then at least one of the two points t_ℓ^n and $t_{\ell+1}^n$ belongs to $S_{j,k}$. Since two consecutive points of the partition are separated by at least $\underline{\pi^n}$, the number of partition points of π^n contained in $S_{j,k}$ is bounded above by

$$\frac{R2^{-j}}{\underline{\pi^n}} + 2.$$

Since the sequence of partitions is balanced, we have

$$\#\{\ell : k \in K_{j,n,\ell}\} \le 2\,\#\{\ell : t_\ell^n \in S_{j,k}\} \le 2\left(\frac{cR2^{-j}}{|\pi^n|} + 2\right) \le (2cR+4)\max\left\{\frac{2^{-j}}{|\pi^n|}, 1\right\}.$$

Set $C_{p,\psi,R,c} := C_\psi^p\big(2(R+1)\big)^{p-1}(2cR+4)$. Then, we obtain

$$\begin{aligned}\sum_{\ell=0}^{N(\pi^n)-1}\left|X_j(t_{\ell+1}^n)-X_j(t_\ell^n)\right|^p &\le C_{p,\psi,R,c}\max\left\{\frac{2^{-j}}{|\pi^n|},1\right\}\min\left\{2^j|\pi^n|,1\right\}^p\sum_{k\in\mathbb{Z}}|c_{j,k}|^p\\ &\le C_{p,\psi,R,c}\min\left\{\left(2^j|\pi^n|\right)^{p-1},1\right\}\sum_{k\in\mathbb{Z}}|c_{j,k}|^p, \qquad (22)\end{aligned}$$

since $\min\{a,1\}^p\max\{\frac{1}{a},1\}\le\min\{1,a^{p-1}\}$ for every $a>0$. We note that for all $j\le J_\beta^\pi(n)$, $\min\left\{\left(2^j|\pi^n|\right)^{p-1},1\right\}\le\left(2^j|\pi^n|\right)^{p-1}$. Hence,

$$\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|X_j(t_{\ell+1}^n)-X_j(t_\ell^n)\right|^p\right)^{1/p}\lesssim|\pi^n|^{1-\frac{1}{p}}2^{j(1-\frac{1}{p})}\left(\sum_k|c_{j,k}|^p\right)^{1/p},$$

where the implicit constant depends only on p, ψ, R and the balance constant c.

Let us now sum over the scales j. For every j and ℓ we set

$$a_{j,\ell}=X_j(t_{\ell+1}^n)-X_j(t_\ell^n).$$

Then

$$\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|\sum_{j=0}^{J_\beta^\pi(n)}\big(X_j(t_{\ell+1}^n)-X_j(t_\ell^n)\big)\right|^p\right)^{1/p}=\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|\sum_{j=0}^{J_\beta^\pi(n)}a_{j,\ell}\right|^p\right)^{1/p}.$$

For each fixed j, the sequence $(a_{j,\ell})_\ell$ can be seen as an element of ℓ^p. Minkowski's inequality in ℓ^p gives

$$\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|\sum_{j=0}^{J_\beta^\pi(n)}a_{j,\ell}\right|^p\right)^{1/p}\le\sum_{j=0}^{J_\beta^\pi(n)}\left(\sum_{\ell=0}^{N(\pi^n)-1}|a_{j,\ell}|^p\right)^{1/p}.$$

Therefore,

$$\begin{aligned}\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|\sum_{j=0}^{J_\beta^\pi(n)}\big(X_j(t_{\ell+1}^n)-X_j(t_\ell^n)\big)\right|^p\right)^{1/p} &\le\sum_{j=0}^{J_\beta^\pi(n)}\left(\sum_{\ell=0}^{N(\pi^n)-1}\left|X_j(t_{\ell+1}^n)-X_j(t_\ell^n)\right|^p\right)^{1/p}\\ &\lesssim|\pi^n|^{1-\frac{1}{p}}\sum_{j=0}^{J_\beta^\pi(n)}2^{j(1-\frac{1}{p})}\left(\sum_k|c_{j,k}|^p\right)^{1/p}\end{aligned}$$

We will now show that the contribution of the scales $j>J_\beta^\pi(n)$ goes to zero as $n\to\infty$. We denote

$$R_{n,\beta}(t):=\sum_{j>J_\beta^\pi(n)}\sum_{k\in\mathbb{Z}}c_{j,k}\psi(2^jt-k).$$

Since the zero extension of $y=x-\tilde{x}$ belongs to $C^\alpha(\mathbb{R})$ and ψ has at least one vanishing moment, from (8) we get $|c_{j,k}|\lessapprox 2^{-j\alpha}$. Moreover, since ψ is compactly supported, for every fixed t and j only a bounded number of wavelets $\psi(2^jt-k)$ are non-zero. Therefore,

$$|R_{n,\beta}(t)|\lesssim\sum_{j>J_\beta^\pi(n)}2^{-\alpha j}\lesssim 2^{-\alpha J_\beta^\pi(n)}.$$

Since the intervals of π^n cover $[0,1]$, we have $1 \geq N(\pi^n)\,\underline{\pi^n}$. Using furthermore the balanced condition of the partition sequence, it follows that

$$\left(\sum_{\ell=0}^{N(\pi^n)-1} |R_{n,\beta}(t^n_{\ell+1}) - R_{n,\beta}(t^n_\ell)|^p\right)^{1/p} \lesssim N(\pi^n)^{1/p} 2^{-\alpha J^\pi_\beta(n)} \leq \left(\frac{c}{|\pi^n|}\right)^{1/p} 2^{-\alpha J^\pi_\beta(n)}.$$

From the definition of $J^\pi_\beta(n)$, we know that $2^{-\beta(J^\pi_\beta(n)+1)} < |\pi^n|$, so that

$$\left(\sum_{\ell=0}^{N(\pi^n)-1} |R_{n,\beta}(t^n_{\ell+1}) - R_{n,\beta}(t^n_\ell)|^p\right)^{1/p} \lesssim 2^{-(\alpha-\beta/p)J^\pi_\beta(n)}$$

where the implicit constant is independent of $\beta \in (0,1)$. Since $\alpha > \beta/p$ by assumption, the contribution of the scales $j > J^\pi_\beta(n)$ tends to zero as $n \to \infty$.

Combining the low-frequency part, the scales $j \leq J^\pi_\beta(n)$ and the remainder $j > J^\pi_\beta(n)$, Minkowski's inequality gives

$$\begin{aligned}[x]^{(p)}_{\pi^n}(1)^{1/p} \lesssim{}& \big(|x(1)-x(0)| + \|g'\|_{L^\infty}\big)|\pi^n|^{1-\frac{1}{p}} \\ &+ |\pi^n|^{1-\frac{1}{p}} \sum_{j=0}^{J^\pi_\beta(n)} 2^{j(1-\frac{1}{p})} \left(\sum_k |c_{j,k}|^p\right)^{1/p} + 2^{-(\alpha-\beta/p)J^\pi_\beta(n)}.\end{aligned}$$

Since $\alpha > \frac{\beta}{p}$, we finally obtain

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n} \lesssim \limsup_{n\to\infty} \eta^{\pi,(p)}_{n,\beta}.$$

□

7.2 Proof of Proposition 3.4

Proof of Proposition 3.4. Recall the notation $A^{(p)}_j$ from (10) and

$$F_\beta(J) = 2^{-\beta(1-\frac{1}{p})J} \sum_{j=0}^{J} 2^{j(1-\frac{1}{p})} A^{(p)}_j, \quad J \geq 0.$$

By the definition of $J^\pi_\beta(n)$,

$$|\pi^n|^{p-1} \asymp 2^{-\beta J^\pi_\beta(n)(p-1)},$$

with constants independent of n. Therefore,

$$\left(\eta^{\pi,(p)}_{n,\beta}\right)^{1/p} \asymp F_\beta(J^\pi_\beta(n)),$$

Similarly,

$$\left(\eta^{\sigma,(p)}_{m,\beta}\right)^{1/p} \asymp F_\beta(J^\sigma_\beta(m)).$$

Assume that $M_\sigma := \sup_{m\geq 0} F_\beta\big(J^\sigma_\beta(m)\big) < \infty$. Let $n \geq 0$ and set

$$J := J^\pi_\beta(n).$$

By assumption, there exists $m \geq 0$ such that

$$J \leq J' := J^\sigma_\beta(m) \leq J + K.$$

Since all the terms in the sum defining F_β are non-negative, we have

$$\sum_{j=0}^{J} 2^{j(1-\frac{1}{p})} A_j^{(p)} \le \sum_{j=0}^{J'} 2^{j(1-\frac{1}{p})} A_j^{(p)}.$$

Moreover,

$$2^{-\beta(1-\frac{1}{p})J} = 2^{\beta(1-\frac{1}{p})(J'-J)} 2^{-\beta(1-\frac{1}{p})J'} \le 2^{\beta(1-\frac{1}{p})K} 2^{-\beta(1-\frac{1}{p})J'}.$$

Therefore,

$$F_\beta(J) \le 2^{\beta(1-\frac{1}{p})K} F_\beta(J') \le 2^{\beta(1-\frac{1}{p})K} M_\sigma.$$

Since n was arbitrary, it follows that

$$\sup_{n\ge 0} F_\beta\big(J_\beta^\pi(n)\big) \le 2^{\beta(1-\frac{1}{p})K} \sup_{m\ge 0} F_\beta\big(J_\beta^\sigma(m)\big).$$

Applying the same argument with π and σ interchanged gives the conclusion. □

7.3 Proof of Proposition 3.7

Proof of Proposition 3.7. Since $|\mathbb{T}^n| = 2^{-n}$, one has $J_\beta^{\mathbb{T}}(n) = \left\lfloor \frac{n}{\beta} \right\rfloor$. Hence, we have

$$\left(\eta_{n,\beta}^{\mathbb{T},(p)}\right)^{1/p} \asymp 2^{-\beta(1-\frac{1}{p})J_\beta^{\mathbb{T}}(n)} \sum_{j=0}^{J_\beta^{\mathbb{T}}(n)} 2^{(1-\frac{1}{p})j} A_j^{(p)}$$

where $A_j^{(p)}$ is defined in (10). Note that

$$J_\beta^{\mathbb{T}}(n+1) - J_\beta^{\mathbb{T}}(n) \le \frac{1}{\beta} + 1.$$

Hence, for every $J \ge 0$, there exists n such that

$$J \le J_\beta^{\mathbb{T}}(n) \le J + \frac{1}{\beta} + 1.$$

By the argument used in the proof of Proposition 3.4, for every J and the above choice of n, we have

$$F_\beta(J) \le 2^{(1+\beta)(1-\frac{1}{p})} F_\beta(J_\beta^{\mathbb{T}}(n)).$$

The reverse inequality between the corresponding suprema is immediate. Thus

$$\sup_{n\ge 0} F_\beta(J_\beta^{\mathbb{T}}(n)) < \infty \qquad \Longleftrightarrow \qquad \sup_{J\ge 0} F_\beta(J) < \infty,$$

where

$$F_\beta(J) = 2^{-\beta(1-\frac{1}{p})J} \sum_{j=0}^{J} 2^{(1-\frac{1}{p})j} A_j^{(p)}.$$

By Lemma 3.6, this is equivalent to $\sup_{J\ge 0} 2^{(1-\beta)(1-\frac{1}{p})J} A_J^{(p)} < \infty$. Since

$$(1-\beta)\left(1-\frac{1}{p}\right) = s_\beta - \frac{1}{p},$$

we obtain

$$\sup_{n\ge 0} \eta_{n,\beta}^{\mathbb{T},(p)} < \infty \qquad \Longleftrightarrow \qquad \sup_{j\ge 0} 2^{j(s_\beta - \frac{1}{p})} A_j^{(p)} < \infty.$$

This is exactly the classical wavelet characterization of $B_{p,\infty}^{s_\beta}(\mathbb{R})$ with the L^∞ normalization used here [14, Chapter 6, Section 10, (10.1)–(10.2)]. The equivalence with $x \in B_{p,\infty}^{s_\beta}([0,1])$ follows from the zero-extension observation preceding Lemma 3.6. □

7.4 Proof of Proposition 4.3

Proof of Proposition 4.3. We keep the notation of Section 2. Write $V_{m+1} := [x]^{(p)}_{\pi^{m+1}}(1)$, and denote by $I^m_\ell = [t^m_\ell, t^m_{\ell+1}]$ the intervals of π^m. For any interval $I \subset [0,1]$ and function f, write $\Delta_I f := f(\sup I) - f(\inf I)$. We also set

$$S_{j,k} := [k2^{-j}, (k+R)2^{-j}] \supset \operatorname{supp}(\psi_{j,k}).$$

By the balanced and finitely refining assumptions, there exist constants $c > 1$ and $M \geq 1$, independent of m, such that

$$|I^m_\ell| \geq \frac{|\pi^m|}{c}, \qquad \#\{J \in \pi^{m+1} : J \subset I^m_\ell\} \leq M.$$

We will use the following two counting facts, valid for every interval $I \subset \mathbb{R}$:

$$\#\{\ell : I^m_\ell \cap I \neq \emptyset\} \leq \frac{c|I|}{|\pi^m|} + 2, \tag{23}$$

$$\#\{k : S_{j,k} \cap I \neq \emptyset\} \leq 2^j|I| + R + 1. \tag{24}$$

All constants below depend only on p, the wavelet, and the constants of balance and complete refinement. We divide the proof into four steps.

Step 1: Decomposition into successive interpolations. Let x_m be the piecewise linear interpolation of x along π^m, and set $g_m := x_{m+1} - x_m$. Since $\pi^0 = \{0,1\}$, we have $x_0 = \tilde{x}$, where $\tilde{x}$ is the affine function joining $x(0)$ and $x(1)$. Hence

$$y := x - \tilde{x} = \sum_{m \geq 0} g_m,$$

with uniform convergence on $[0,1]$. Each g_m vanishes at the endpoints and is extended by zero outside $[0,1]$. For $j \geq 0$, $k \in \mathbb{Z}$, define

$$c^{(m)}_{j,k} := 2^j \int_{\mathbb{R}} g_m(t)\psi(2^j t - k)\, dt, \qquad A^{(p)}_{j,m} := \Big(\sum_k |c^{(m)}_{j,k}|^p\Big)^{1/p}.$$

Since $\sum_{m=0}^{N-1} g_m = x_N - \tilde{x}$, we have

$$\begin{aligned}\left|c_{j,k} - \sum_{m=0}^{N-1} c^{(m)}_{j,k}\right| &= 2^j \left|\int_{\mathbb{R}} (x(t) - x_N(t))\psi(2^j t - k)\, dt\right| \\ &\leq \|\psi\|_{L^1}\|x - x_N\|_\infty \xrightarrow[N\to\infty]{} 0,\end{aligned}$$

where $x - x_N$ is extended by zero outside $[0,1]$. Consequently,

$$c_{j,k} = \sum_{m \geq 0} c^{(m)}_{j,k}. \tag{25}$$

For each fixed j, only finitely many k satisfy $S_{j,k} \cap [0,1] \neq \emptyset$. For every $N \geq 0$, using the triangle inequality in ℓ^p and Minkowski's inequality for the finite sum, we obtain

$$\begin{aligned}A^{(p)}_j := \Big(\sum_k |c_{j,k}|^p\Big)^{1/p} &\leq \Big(\sum_k \Big|\sum_{0\leq m\leq N} c^{(m)}_{j,k}\Big|^p\Big)^{1/p} + \Big(\sum_k \Big|\sum_{m>N} c^{(m)}_{j,k}\Big|^p\Big)^{1/p} \\ &\leq \sum_{0\leq m\leq N} \Big(\sum_k \Big|c^{(m)}_{j,k}\Big|^p\Big)^{1/p} + \Big(\sum_k \Big|2^j \int_{\mathbb{R}} (x - x_{N+1})(t)\psi_{j,k}(t)\, dt\Big|^p\Big)^{1/p}\end{aligned}$$

$$\leq \sum_{m\geq 0} A_{j,m}^{(p)} + (2^j + R + 1)^{\frac{1}{p}} \|\psi\|_{L^1} \|x - x_{N+1}\|_\infty.$$

Letting $N \to \infty$ and using $\|x - x_{N+1}\|_\infty \to 0$ proves

$$A_j^{(p)} \leq \sum_{m\geq 0} A_{j,m}^{(p)}. \tag{26}$$

Step 2: Control on each parent interval. Since $g_m = x_{m+1} - x_m$ is continuous, piecewise linear on the intervals of π^{m+1}, and vanishes at the points of π^m, its maximum on each interval of π^m is attained at some point of π^{m+1}. Following the notation $I_{k'}^m = [t_{k'}^m, t_{k'+1}^m]$, and using Hölder's inequality, we obtain

$$\begin{aligned}\max_{t\in I_{k'}^m} |g_m(t)| &\leq 2 \sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} \left|x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})\right| \\ &\leq 2\lfloor cb\rfloor^{1-\frac{1}{p}} \left(\sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} \left|x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})\right|^p \right)^{1/p}. \end{aligned} \tag{27}$$

Here b and c are the constants from (2) and (4).

Step 3: Estimating the coefficients at the two scales. Instead of bounding $c_{j,k}$ directly, we now bound $c_{j,k}^{(m)}$ from (25) instead. Now similar to $J_1(n)$ as defined in Theorem 3.1, we set $n = n(j) := \max\{m : |\pi^m| \geq 2^{-j}\}$ as in Proposition 4.3.

Case 1: $m > n$. Since $m > n$, $|\pi^m| \leq |\pi^{n+1}| < 2^{-j} \leq |\pi^n|$. So we get,

$$\begin{aligned}|c_{j,k}^{(m)}| &= \left|2^j \int_{\mathbb{R}} (x_{m+1}(t) - x_m(t))\psi_{j,k}(t)\,dt\right| \\ &\leq 2^j \|\psi\|_\infty \int_{\operatorname{supp}\psi_{j,k}} |x_{m+1}(t) - x_m(t)|\,dt \\ &= 2^j \|\psi\|_\infty \sum_{\{k'\,:\,\operatorname{supp}\psi_{j,k}\cap I_{k'}^m \neq \emptyset\}} \int_{\operatorname{supp}\psi_{j,k}\cap I_{k'}^m} |x_{m+1}(t) - x_m(t)|\,dt \\ &\overset{\text{Step 2}}{\lessapprox} 2^j \|\psi\|_\infty \sum_{\{k'\,:\,\operatorname{supp}\psi_{j,k}\cap I_{k'}^m \neq \emptyset\}} |\pi^m| \left(2\lfloor cb\rfloor^{1-\frac{1}{p}} \Big(\sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} |x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})|^p \Big)^{1/p} \right). \end{aligned}$$

Now notice that from (23),

$$\#\{k' \,:\, \operatorname{supp}\psi_{j,k} \cap I_{k'}^m \neq \emptyset\} \leq \frac{cR2^{-j}}{|\pi^m|} + 2 \lessapprox \frac{2^{-j}}{|\pi^m|},$$

where the last inequality follows from $|\pi^m| < 2^{-j}$. So combining and taking a sum over k we get:

$$\begin{aligned}\sum_k |c_{j,k}^{(m)}|^p &\lessapprox 2^{pj} \sum_k \left(\sum_{\{k'\,:\,\operatorname{supp}\psi_{j,k}\cap I_{k'}^m \neq \emptyset\}} |\pi^m| \Big(\sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} |x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})|^p \Big)^{1/p} \right)^p \\ &\overset{\text{Hölder}}{\lessapprox} 2^{pj} \Big(\frac{2^{-j}}{|\pi^m|}\Big)^{p-1} \sum_k \sum_{\{k'\,:\,\operatorname{supp}\psi_{j,k}\cap I_{k'}^m \neq \emptyset\}} |\pi^m|^p \Big(\sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} |x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})|^p \Big) \\ &= 2^j |\pi^m| \sum_k \sum_{\{k'\,:\,\operatorname{supp}\psi_{j,k}\cap I_{k'}^m \neq \emptyset\}} \Big(\sum_{\{\ell\,:\, I_\ell^{m+1}\subset I_{k'}^m\}} |x(t_{\ell+1}^{m+1}) - x(t_\ell^{m+1})|^p \Big). \end{aligned}$$

Moreover, for every fixed parent interval $I^m_{k'}$, (24) gives

$$\#\{k : \operatorname{supp}\psi_{j,k} \cap I^m_{k'} \neq \emptyset\} \le 2^j|\pi^m| + R + 1 \le R + 2,$$

since $|\pi^m| < 2^{-j}$. Therefore, exchanging the sums over k and k', we get

$$\begin{aligned}\sum_k |c^{(m)}_{j,k}|^p &\lessapprox 2^j|\pi^m| \sum_{k'} \Big(\sum_{\{\ell : I^{m+1}_\ell \subset I^m_{k'}\}} |x(t^{m+1}_{\ell+1}) - x(t^{m+1}_\ell)|^p \Big) \\ &= 2^j|\pi^m| \sum_\ell |x(t^{m+1}_{\ell+1}) - x(t^{m+1}_\ell)|^p \\ &= 2^j|\pi^m| [x]^{(p)}_{\pi^{m+1}}(1).\end{aligned}$$

Finally, note that in the above calculation the $\lessapprox$ involves a constant which is independent of m and j. Additionally, since π is complete refining, using the constant a from (4), we have $|\pi^m| \le (1+a)^{n-m}|\pi^n|$. So we get

$$\begin{aligned}\sum_{m>n} A^{(p)}_{j,m} &\lessapprox \sum_{m>n} (2^j(1+a)^{n-m}|\pi^n|)^{\frac{1}{p}} \Big([x]^{(p)}_{\pi^{m+1}}(1) \Big)^{\frac{1}{p}} \\ &\lessapprox (2^j|\pi^n|)^{\frac{1}{p}} \sum_{m>n} \Big((1+a)^{n-m} [x]^{(p)}_{\pi^{m+1}}(1) \Big)^{\frac{1}{p}} \\ &\lessapprox \sum_{m>n} (1+a)^{(n-m)\frac{1}{p}} \big([x]^{(p)}_{\pi^{m+1}}(1) \big)^{\frac{1}{p}} \qquad (28)\end{aligned}$$

where the last inequality follows from $2^j|\pi^n| < b$.

Case 2: $m \le n$. In this case, from the construction of $n = n(j)$, we have $2^{-j} \le |\pi^n| \le |\pi^m| \le b|\pi^{m+1}|$. Using the vanishing moment of ψ, for any fixed $s \in S_{j,k}$ we have

$$c^{(m)}_{j,k} = 2^j \int_{\mathbb{R}} (g_m(t) - g_m(s)) \psi(2^j t - k)\, dt.$$

So $|c^{(m)}_{j,k}| \le \|\psi\|_{L^1} \operatorname{osc}_{S_{j,k}}(g_m)$. On each I^{m+1}_ℓ, the function g_m is linear with slope $|\Delta_{I^{m+1}_\ell} g_m|/|I^{m+1}_\ell| \le c\,|\Delta_{I^{m+1}_\ell} g_m|/|\pi^{m+1}|$ using the balanced property of the partition sequence, and $|S_{j,k}| = R2^{-j}$; hence, with $L_{j,k} := \{\ell : I^{m+1}_\ell \cap S_{j,k} \neq \emptyset\}$, for any $s < t$ in $S_{j,k}$ we have

$$|g_m(t) - g_m(s)| \le \sum_{\ell \in L_{j,k}} \frac{|\Delta_{I^{m+1}_\ell} g_m|}{|I^{m+1}_\ell|} |[s,t] \cap I^{m+1}_\ell| \le \frac{cR\,2^{-j}}{|\pi^{m+1}|} \sum_{\ell \in L_{j,k}} |\Delta_{I^{m+1}_\ell} g_m|.$$

Taking the supremum over $s, t \in S_{j,k}$ gives the same bound for $\operatorname{osc}_{S_{j,k}}(g_m)$. Moreover, by (23),

$$\#L_{j,k} \le \frac{cR\,2^{-j}}{|\pi^{m+1}|} + 2 \le cRb + 2.$$

By Hölder's inequality, it follows that

$$\begin{aligned}|c^{(m)}_{j,k}|^p &\le C \Big(\frac{2^{-j}}{|\pi^{m+1}|} \Big)^p \Big(\sum_{\ell \in L_{j,k}} |\Delta_{I^{m+1}_\ell} g_m| \Big)^p \\ &\overset{\text{Hölder}}{\le} C \Big(\frac{2^{-j}}{|\pi^{m+1}|} \Big)^p (\#L_{j,k})^{p-1} \sum_{\ell \in L_{j,k}} |\Delta_{I^{m+1}_\ell} g_m|^p \\ &\le C' \Big(\frac{2^{-j}}{|\pi^{m+1}|} \Big)^p \sum_{\ell \in L_{j,k}} |\Delta_{I^{m+1}_\ell} g_m|^p.\end{aligned}$$

By (24) and $2^j|\pi^{m+1}| \geq 1/b$ since $m \leq n$, a fixed ℓ belongs to $L_{j,k}$ for at most $2^j|\pi^{m+1}|+R+1 \leq (R+2)b\,2^j|\pi^{m+1}|$ values of k.

Now, for each child $I_\ell^{m+1} \subset I_{k'(\ell)}^m$, since x_m is affine on $I_{k'(\ell)}^m$ and x_{m+1} coincides with x at the endpoints of I_ℓ^{m+1}, we have

$$|\Delta_{I_\ell^{m+1}} g_m| \leq |\Delta_{I_\ell^{m+1}} x| + |\Delta_{I_{k'(\ell)}^m} x|.$$

Since each parent contains at most $\lfloor cb \rfloor$ children, Hölder's inequality gives

$$\sum_\ell |\Delta_{I_{k'(\ell)}^m} x|^p \leq \lfloor cb \rfloor^p [x]^{(p)}_{\pi^{m+1}}(1),$$

and therefore

$$\sum_\ell |\Delta_{I_\ell^{m+1}} g_m|^p \leq C[x]^{(p)}_{\pi^{m+1}}(1).$$

Summing over k and using the previous counting estimate, we get

$$\begin{aligned}
\sum_k |c_{j,k}^{(m)}|^p &\leq C\Big(\frac{2^{-j}}{|\pi^{m+1}|}\Big)^p 2^j|\pi^{m+1}| \sum_\ell |\Delta_{I_\ell^{m+1}} g_m|^p \\
&= C\Big(\frac{2^{-j}}{|\pi^{m+1}|}\Big)^{p-1} \sum_\ell |\Delta_{I_\ell^{m+1}} g_m|^p \\
&\leq C'\big(2^j|\pi^m|\big)^{1-p} [x]^{(p)}_{\pi^{m+1}}(1),
\end{aligned}$$

where the last inequality also uses $|\pi^{m+1}| \geq |\pi^m|/b$.

Since π is complete refining, using the constant from (4) we have, for $m \leq n$, the lower bound $|\pi^m| \geq (1+a)^{n-m}|\pi^n|$. As $1-p<0$ we get

$$\begin{aligned}
\sum_{m\leq n} A_{j,m}^{(p)} = \sum_{m\leq n} \big(\sum_k |c_{j,k}^{(m)}|^p\big)^{1/p} &\lessapprox \sum_{m\leq n} \big(\big(2^j|\pi^m|\big)^{1-p} [x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \\
&\leq \sum_{m\leq n} \big(\big(2^j(1+a)^{(n-m)}|\pi^n|\big)^{1-p} [x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \\
&= (2^j|\pi^n|)^{\frac{1}{p}-1} \sum_{m\leq n} (1+a)^{-(n-m)(1-\frac{1}{p})} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \\
&\leq \sum_{m\leq n} (1+a)^{-(n-m)(1-\frac{1}{p})} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \qquad (29)
\end{aligned}$$

where the last inequality follows from $2^j|\pi^n| \geq 1$ and $\frac{1}{p}-1<0$.

Finally combining (26), (28) and (29) we get,

$$A_j^{(p)} = \Big(\sum_k |c_{j,k}|^p\Big)^{1/p} \leq \sum_{m\leq n} A_{j,m}^{(p)} + \sum_{m>n} A_{j,m}^{(p)} \lessapprox \sum_{m\geq 0} \rho^{|m-n(j)|} \Big([x]^{(p)}_{\pi^{m+1}}(1)\Big)^{1/p},$$

where ρ is defined in (15). This completes the proof of (16).

Step 4: The asymptotic estimate. Let $L := \limsup_{m\to\infty} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p}$. If $L=\infty$, (17) is immediate. Otherwise, fix $\varepsilon > 0$. By definition of the limsup, there exists M_0 such that $\big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \leq L+\varepsilon$ for every $m \geq M_0$. From (16), we have

$$A_j^{(p)} \leq C_1 \sum_{m<M_0} \rho^{|m-n(j)|} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} + C_1 \sum_{m\geq M_0} \rho^{|m-n(j)|} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p}.$$

First, since $n(j) \to \infty$ as $j \to \infty$, for every fixed m we have $\rho^{|m-n(j)|} \to 0$. As the first sum contains only finitely many terms, we get

$$\lim_{j\to\infty} \sum_{m<M_0} \rho^{|m-n(j)|} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} = 0.$$

For the second sum, using $\big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \le L + \varepsilon$ for $m \ge M_0$, we obtain

$$\sum_{m\ge M_0} \rho^{|m-n(j)|} \big([x]^{(p)}_{\pi^{m+1}}(1)\big)^{1/p} \le (L+\varepsilon) \sum_{m\ge 0} \rho^{|m-n(j)|}.$$

Moreover,

$$\sum_{m\ge 0} \rho^{|m-n(j)|} \le 1 + 2\sum_{r\ge 1} \rho^r = \frac{1+\rho}{1-\rho} < \frac{2}{1-\rho}.$$

Combining these estimates and taking the limsup as $j \to \infty$, we get

$$\limsup_{j\to\infty} A_j^{(p)} \le \frac{2C_1}{1-\rho}(L+\varepsilon).$$

Letting $\varepsilon \downarrow 0$ and raising to the power p proves (17). □

7.5 Proof of Proposition 5.1

Proof of Proposition 5.1. In the proof of Theorem 3.1 up to (22) we have never used the relationship between j, n. So for j-th wavelet level of x, denoted by X_j we get

$$\sum_{\ell=0}^{N(\pi^n)-1} \big|X_j(t^n_{\ell+1}) - X_j(t^n_\ell)\big|^p \le C_{p,\psi,R,c} \min\big\{(2^j|\pi^n|)^{p-1}, 1\big\} \Big(A_j^{(p)}\Big)^p,$$

for all n, j. Now, likewise to the proof of Theorem 3.1, apply Minkowski's inequality to

$$x^m := \tilde{x} + g + \sum_{j\le m} X_j.$$

Since $\sum_j A_j^{(p)} < \infty$, the wavelet series converges uniformly and

$$\|x - x^m\|_\infty \le (R+1)\|\psi\|_\infty \sum_{j>m} A_j^{(p)} \longrightarrow 0.$$

The affine and low-frequency terms satisfy $([\tilde{x}+g]^{(p)}_{\pi^n}(1))^{1/p} \le C_0|\pi^n|^{1-1/p}$. Hence, passing to the limit as $m \to \infty$, we obtain

$$\big([x]^{(p)}_{\pi^n}(1)\big)^{1/p} \le C_0|\pi^n|^{1-1/p} + C\sum_{j\ge 0} \min\big\{(2^j|\pi^n|)^{1-\frac{1}{p}}, 1\big\} A_j^{(p)}.$$

The additional term $C_0|\pi^n|^{1-1/p}$ tends to zero as $n \to \infty$. Now, under the assumption of $\sum_{j\ge 0} A_j^{(p)} < \infty$, which is independent of the sequence of partitions π, given any $\epsilon > 0$ there exists N_0 such that $\sum_{j>N_0} A_j^{(p)} < \epsilon$. So

$$\begin{aligned}\limsup_n \big([x]^{(p)}_{\pi^n}(1)\big)^{1/p} &\le \limsup_n C\sum_{j\ge 0} \min\big\{(2^j|\pi^n|)^{1-\frac{1}{p}}, 1\big\} A_j^{(p)} \\ &\le C\limsup_n \sum_{j\le N_0} \min\big\{(2^j|\pi^n|)^{1-\frac{1}{p}}, 1\big\} A_j^{(p)} + C\limsup_n \sum_{j>N_0} A_j^{(p)}\end{aligned}$$

$$\leq C \sum_{j \leq N_0} (\limsup_n \min\left\{(2^j|\pi^n|)^{1-\frac{1}{p}}, 1\right\}) A_j^{(p)} + C\epsilon \leq C\epsilon.$$

Since the above argument is true for all $\epsilon > 0$ and for any balanced sequence of partitions with vanishing mesh π, we get

$$\sum_{j \geq 0} A_j^{(p)} < \infty \quad \Longrightarrow \quad x \in V_\pi^p \subset \mathcal{X}_\pi^p \text{ and } [x]_\pi^{(p)} \equiv 0 \text{ for all balanced } \pi. \tag{30}$$

For (ii) let π be balanced and complete refining, $p > 1$, and $x \in \mathcal{X}_\pi^p \cap C^\alpha([0,1])$ for some $\alpha > 0$. Fix $0 < \alpha_0 < \min\{\alpha, 1\}$. For all $r > p$ we first use Hölder inequality and then use the bound (8) to get,

$$\left(A_j^{(r)}\right)^r \leq \left(\sup_k |c_{j,k}|\right)^{r-p} \left(A_j^{(p)}\right)^p \lessapprox \left(\sup_{i \geq 0} A_i^{(p)}\right)^p 2^{-\alpha_0(r-p)j}.$$

By Proposition 4.3, $\sup_j A_j^{(p)} < \infty$, so $A_j^{(r)} \leq C\, 2^{-\alpha_0(1-\frac{p}{r})j}$ and $\sum_{j \geq 0} A_j^{(r)} < \infty$.

Combining with (30) the proof is complete. □

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